Differentiable Transient Rendering

Supplemental Document

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1 MATHEMATICAL BACKGROUND

In differentiable rendering, the image can be represented as an integral over an evolving manifold containing the path space, and s cene derivatives can be evaluated using transport theorems. For completeness, we introduce here the related mathematical background. We will follow existing terminology and definitions of fluid and continuum mechanics [Cermelli et al. 2005], mathematics [Seguin and Fried 2014], and their application to differentiable rendering [Zhang et al. 2020].

1.1 Mathematical Notions

We first add details of common concepts in mathematical analysis and differential geometry used in our work.

Open and closed sets. A set in the Euclidean space $U \subset \mathbb{R}^n$ is called an *open set* if for any $\mathbf{x} \in U$ there exists $\epsilon > 0$ s.t. $\{\mathbf{y} \in \mathbb{R}^n \mid \|\mathbf{y} - \mathbf{x}\| < \epsilon\} \subset U$. A set $U \subset X \subset \mathbb{R}^n$ is called an *open set relative to* X if for any $\mathbf{x} \in U$ there exists $\epsilon > 0$ s.t. $\{\mathbf{y} \in X \mid \|\mathbf{y} - \mathbf{x}\| < \epsilon\} \subset U$. A set $A \subset \mathbb{R}^n$ (resp. $A \subset X \subset \mathbb{R}^n$) is called a *closed set* (resp. *closed set relative to* X) if $\mathbb{R}^n - A$ (resp. $X - A$) is an open set (resp. open set relative to X).

Continuous functions. For sets X and $Y \subset \mathbb{R}^n$, a function $\varphi : X \to Y$ is called *continuous* if for any open set U relative to Y, $\varphi^{-1}(U)$ is open relative to X. Note that this definition is equivalent to the epsilon-delta argument.

Manifolds. We define the halfspace as:

$$\mathbb{H}^n := \left\{ (x_1 ... x_n) \in \mathbb{R}^n \mid x_1 \geq 0 \right\}. \tag{1}$$

Then a set $\mathcal{M} \subset \mathbb{R}^n$ is called an *m-dimensional manifold* (or *m*-manifold) if for any $\mathbf{x} \in \mathcal{M}$ there exists an open set $U_\mathbf{x}$ relative to $\mathcal{M}$ and a one-to-one function $\phi_\mathbf{x} : U_\mathbf{x} \to \phi_\mathbf{x}(U_\mathbf{x}) \subset \mathbb{H}^m$, called a chart, such that both $\phi_\mathbf{x}$ and $\phi_\mathbf{x}^{-1}$ are continuous. Note that if we can choose each chart $\phi_\mathbf{x}$ so that both $\phi_\mathbf{x}$ and $\phi_\mathbf{x}^{-1}$ are C^i-differentiable then $\mathcal{M}$ is called a C^i-differentiable manifold. The *boundary* (or boundary manifold) of $\mathcal{M}$ is defined as:

$$\partial\mathcal{M} := \{\mathbf{x} \in \mathcal{M} \mid \text{ the first coordinate of } \phi_\mathbf{x}(\mathbf{x}) \text{ is zero}\}. \tag{2}$$

Note that this definition is independent of the choice of a particular open set $U_\mathbf{x}$ and chart $\phi_\mathbf{x}$, and that $\partial\mathcal{M}$ is a $(m-1)$-manifold. The *interior* of $\mathcal{M}$ is defined as $\mathrm{Int}(\mathcal{M}) := \mathcal{M} - \partial\mathcal{M}$.

We can intuitively understand an *m*-manifold $\mathcal{M}$ as a set of space of points that need at least m real numbers to parameterize all points in $\mathcal{M}$. For instance, a surface mesh embedded in $\mathbb{R}^3$ is a 2-manifold, and the space of light paths Ω_k with $k+1$ vertices is a $2(k+1)$-manifold embedded in $\mathbb{R}^{3(k+1)}$.

Tangent space. Suppose that $\mathcal{M} \subset \mathbb{R}^n$ is a C^1-differentiable *m*-manifold. The *tangent space* of $\mathcal{M}$ on $\mathbf{x} \in \mathcal{M}$ is then defined as:

$$\begin{aligned} T_\mathbf{x}\mathcal{M} := \{\dot{\gamma}(0) \mid & \gamma : (-\epsilon, \epsilon) \to \mathcal{M} \text{ is a differentiable curve,} \\ & \text{and } \gamma(0) = \mathbf{x}\} \subset \mathbb{R}^n, \end{aligned} \tag{3}$$

where the derivative (velocity w.r.t. parameterization) of the curve $\dot{\gamma}(0)$ can be evaluated as usual in $\mathbb{R}^n$. Note that $T_\mathbf{x}\mathcal{M}$ is an *m*-dimensional vector space; a vector in $T_\mathbf{x}\mathcal{M}$ is called a *tangent vector*.

1.2 Evolving Manifolds

An *evolving m-manifold* $\mathcal{M}(\boldsymbol{\theta}) \subset \mathbb{R}^n$ with respect to the *scene parameter* $\boldsymbol{\theta} \in \mathbb{R}^d$ can be considered as a function mapping each value of the parameter $\boldsymbol{\theta}$ to each *m*-manifold. Its *trajectory* is defined as $\mathcal{J} := \left\{ (\mathbf{x}, \boldsymbol{\theta}) \mid x \in \mathcal{M}(\boldsymbol{\theta}), \boldsymbol{\theta} \in \mathbb{R}^d \right\} \subset \mathbb{R}^{n+d}$. When $\mathcal{M}(\boldsymbol{\theta})$ evolves continuously, we can assume that $\mathcal{J}$ is an $(m+d)$-manifold. In the following we fix the scene parameters vector $\boldsymbol{\theta} = (\theta_1 ... \theta_d)$ as a single scalar θ for simplicity. Generalization to vector $\boldsymbol{\theta}$ will be introduced at the end of this section.

While we have the motion of the entire manifold $\mathcal{M}$ with respect to θ, describing the motion of a single point $\mathbf{x} \in \mathcal{M}(\theta)$ cannot be defined in a trivial way. A *local parameterization* is defined as a one-to-one function $\hat{\mathbf{x}} : U \to \mathcal{J}$ such that U is open in $\mathbb{R}^n \times \mathbb{R}$ and $\hat{\mathbf{x}}(\mathbf{p}, \theta') \in \mathcal{M}(\theta')$ for any $(\mathbf{p}, \theta') \in U$. Suppose a local parameterization for a given point $\mathbf{x} \in \mathcal{M}(\theta)$, i.e., there exists $(\mathbf{p}_0, \theta) \in U$ s.t. $\hat{\mathbf{x}}(\mathbf{p}_0, \theta) = \mathbf{x}$. Then we can define the *local velocity* of $\mathbf{x}$ as:

$$v(\mathbf{x}, \theta) := \left. \frac{\partial}{\partial\theta'} \hat{\mathbf{x}}(\mathbf{p}_0, \theta') \right|_{\theta'=\theta} \in \mathbb{R}^n. \tag{4}$$

The local velocity depends on the choice of local parameterization. Unlike fluid or continuum mechanics, we should eliminate this dependency to get well-defined formulations on evolving manifolds. When the codimenison of $\mathcal{M}$ is one, i.e., $n = m + 1$, the *scalar normal velocity* $\mathcal{V}_\mathcal{M}(\mathbf{x}, \theta)$ and *local tangential velocity* $v_{\tan}(\mathbf{x}, \theta)$ of a given point $\mathbf{x} \in \mathcal{M}(\theta)$ are defined as:

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$$\mathcal{V}_{\mathcal{M}}(\mathbf{x},\theta) := v(\mathbf{x},\theta)\cdot\mathbf{n}(x,\theta), \tag{5}$$

$$v_{\tan}(\mathbf{x},\theta) := v(\mathbf{x},\theta) - \mathcal{V}_{\mathcal{M}}(x,\theta)\,\mathbf{n}(x,\theta), \tag{6}$$

where $\mathbf{n}(x,\theta)$ denotes the unit normal vector of $\mathcal{M}(\theta)$ at $\mathbf{x}$. Then the local tangential velocity $v_{\tan}(\mathbf{x},\theta)$ still depends on the local parameterization, but the scalar normal velocity $\mathcal{V}_{\mathcal{M}}(\mathbf{x},\theta)$ is independent of such local parameterization. However, while the surface geometry embedded in $\mathbb{R}^3$ has codimension one, the space of light paths Ω_k has a higher codimension $(k+1)$, so the normal vector $\mathbf{n}$ cannot be defined. Thus, more generally, the local tangential velocity and the *vector normal velocity* $\vec{\mathcal{V}}_{\mathcal{M}}(\mathbf{x},\theta)$ are defined as:

$$v_{\tan}(\mathbf{x},\theta) := \Pi_{T_{\mathbf{x}}\mathcal{M}}(v(\mathbf{x},\theta)), \tag{7}$$

$$\vec{\mathcal{V}}_{\mathcal{M}}(\mathbf{x},\theta) := v(\mathbf{x},\theta) - v_{\tan}(\mathbf{x},\theta), \tag{8}$$

where $\Pi_{T_{\mathbf{x}}\mathcal{M}}\colon \mathbb{R}^n \to T_{\mathbf{x}}\mathcal{M}$ denotes the canonical projection (orthogonal projection) from a vector space onto a subspace. Note that the definitions of $v_{\tan}$ in Equations (6) and (7) are equivalent, while $\mathcal{V}_{\mathcal{M}}$ in Equation (5) and $\vec{\mathcal{V}}_{\mathcal{M}}$ in Equation (8) are related as $\vec{\mathcal{V}}_{\mathcal{M}} = \mathcal{V}_{\mathcal{M}}\mathbf{n}$ when $n = m+1$. Steady-state path-space differentiable rendering [Zhang et al. 2020] did not use the vector normal velocity since the authors relied on the Reynolds transport relation on a 2D manifold embedded in $\mathbb{R}^3$. This is no longer possible in transient state, an thus the need to apply the generalized transport theorem on the path space $\Omega_k \subset \mathbb{R}^{3(k+1)}$.

The boundary of the evolving manifold, $\partial\mathcal{M}(\theta)$, is also an evolving manifold. We can define a local velocity $v_{\partial\mathcal{M}}(\mathbf{x},\theta)$, a local tangential velocity $v_{\tan,\partial\mathcal{M}}(\mathbf{x},\theta)$, and a vector normal velocity $\vec{\mathcal{V}}_{\partial\mathcal{M}}(\mathbf{x},\theta)$ on $\partial\mathcal{M}(\theta)$ in the same fashion, but the scalar normal velocity $\mathcal{V}_{\partial\mathcal{M}}(\mathbf{x},\theta)$ at $\mathbf{x}\in\partial\mathcal{M}(\theta)$ is defined differently as:

$$v_{\tan,\partial\mathcal{M}}(\mathbf{x},\theta) := \Pi_{T_{\mathbf{x}}\partial\mathcal{M}}\left(v_{\partial\mathcal{M}}(\mathbf{x},\theta)\right), \tag{9}$$

$$\vec{\mathcal{V}}_{\partial\mathcal{M}}(\mathbf{x},\theta) := v_{\partial\mathcal{M}}(\mathbf{x},\theta) - v_{\tan,\partial\mathcal{M}}(\mathbf{x},\theta), \tag{10}$$

$$\mathcal{V}_{\partial\mathcal{M}}(\mathbf{x},\theta) := \vec{\mathcal{V}}_{\partial\mathcal{M}}(\mathbf{x},\theta)\cdot\mathbf{n}_{\partial\mathcal{M}}(\mathbf{x},\theta), \tag{11}$$

where $\mathbf{n}_{\partial\mathcal{M}}(\mathbf{x},\theta)\in T_{\mathbf{x}}\mathcal{M}(\theta)$. Note that when restricting the direction of the unit normal vector $\mathbf{n}_{\partial\mathcal{M}}(\mathbf{x},\theta)$ into $T_{\mathbf{x}}\mathcal{M}(\theta)$ (instead of into $\mathbb{R}^n$) the outgoing normal direction $\mathbf{n}_{\partial\mathcal{M}}$ from $\mathcal{M}(\theta)$ is determined uniquely, so that we can use the scalar normal velocity of $\partial\mathcal{M}(\theta)$ under the general dimensionality of $\mathcal{M}(\theta)$ and $\mathbb{R}^n$.

1.3 Generalized Transport Theorem for Evolving Manifolds

Suppose that there is a scalar field $\varphi\colon \mathcal{J}\to\mathbb{R}$ defined on an evolving manifold. Generally, the field φ could not be defined on the entire space $\mathbb{R}^{n+1}$, so the partial derivative of the field φ with respect to θ cannot be defined in a trivial way. In other words, when trying to evaluate $\lim_{\epsilon\to 0}\frac{\varphi(\mathbf{x},\theta+\epsilon)-\varphi(\mathbf{x},\theta)}{\epsilon}$, the numerator cannot be defined unless $\mathbf{x}$ lies on both $\mathcal{M}(\theta)$ and $\mathcal{M}(\theta+\epsilon)$. Therefore, we first define a derivative which depends on the choice of local parameterization, then we can define a parameterization-independent derivative from the dependent one. The *derivative* $\dot{\varphi}(\mathbf{x},\theta)$ w.r.t. θ, which depends on choice of local parameterization, and the *normal derivative* $\varphi^{\square}(\mathbf{x},\theta)$ w.r.t. θ, which is independent of local parameterization, are defined as follows:

$$\dot{\varphi}(\mathbf{x},\theta) := \left.\frac{\partial}{\partial\theta'}\varphi\left(\hat{\mathbf{x}}\left(\mathbf{p}_0,\theta'\right),\theta'\right)\right|_{\theta'=\theta}, \tag{12}$$

$$\overset{\square}{\varphi}(\mathbf{x},\theta) := \dot{\varphi}(\mathbf{x},\theta) - v_{\tan}(\mathbf{x},\theta)\cdot\mathrm{grad}_{\mathcal{M}}\varphi(\mathbf{x},\theta). \tag{13}$$

Now we investigate the derivative of the integration over evolving manifolds. Seguin et al. [2014] showed that the derivative of the integral of φ over the evolving manifold $\mathcal{M}(\theta)$ can be represented as the *transport theorem for evolving manifolds*:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\mathcal{M}(\theta)}\varphi\mathrm{d}\mu_{\mathcal{M}} = \int_{\mathcal{M}(\theta)}\left(\overset{\square}{\varphi} - \varphi\vec{\kappa}\cdot\vec{\mathcal{V}}_{\mathcal{M}}\right)\mathrm{d}\mu_{\mathcal{M}} + \int_{\partial\mathcal{M}(\theta)}\varphi\mathcal{V}_{\partial\mathcal{M}}\mathrm{d}\mu_{\partial\mathcal{M}}, \tag{14}$$

where $\vec{\kappa}$ is the *total curvature vector*, m times the mean curvature vector on the m-manifold $\mathcal{M}(\theta)$, and $\mu_{\mathcal{M}}$ and $\mu_{\partial\mathcal{M}}$ are the measures on $\mathcal{M}(\theta)$ and $\partial\mathcal{M}(\theta)$, respectively. The mean curvature vector on a m-manifold embedded in $\mathbb{R}^n$ has been defined in differential geometry [Carmo 1992; Chen 1975]. According to their definition, the mean curvature vectors can be well defined for arbitrary codimensions, i.e., even if $n > m+1$, so that the unit normal vector on the manifold is not uniquely defined.

Note that this transport theorem has been given different names depending on the dimension m and the codimension n. For the simplest case, $m = n = 1$, the theorem is called the Leibniz integral rule, and for the case of $m = n = 3$ (or any case of $m = n$) the theorem is usually called the Reynolds transport theorem. Zhang et al. [2020] used this transport theorem for the particular case of $m = 2$ and $n = 3$, and applied it iteratively. Our work generalizes this for any dimension and codimension.

1.4 Generalized Transport Theorem with Discontinuity

In this section we will treat the case when the scalar field φ contains discontinuities. For simplicity, our notations follow previous work [Zhang et al. 2020]. The *discontinuity set* (discontinuity submanifold) of $\mathcal{M}(\theta)$ with respect to φ is defined as:

$$\Delta\mathcal{M}[\varphi](\theta) := \{\mathbf{x}\in\mathcal{M}(\theta) \mid \varphi(\cdot,\theta) \text{ is discontinuous at } \mathbf{x}\}. \tag{15}$$

We assume that $\Delta\mathcal{M}[\varphi](\theta)$ can be represented as a finite union of $(m-1)$-submanifolds of $\mathcal{M}(\theta)$, and that $\Delta\mathcal{M}[\varphi](\theta)$ itself evolves continuously. Then the *continuous interior* $\mathrm{Int}^{\circ}(\mathcal{M})[\varphi](\theta)$ and the *extended boundary* $\partial\overline{\mathcal{M}}[\varphi](\theta)$ of $\mathcal{M}$ with respect to φ are defined as follows:

$$\begin{aligned}\mathrm{Int}^{\circ}(\mathcal{M})[\varphi](\theta) :=& \mathrm{Int}(\mathcal{M})(\theta) - \Delta\mathcal{M}[\varphi](\theta)\\ =& \mathcal{M}(\theta) - \partial\mathcal{M}(\theta) - \Delta\mathcal{M}[\varphi](\theta),\end{aligned} \tag{16}$$

$$\partial\overline{\mathcal{M}}[\varphi](\theta) := \partial\mathcal{M}(\theta)\cup\Delta\mathcal{M}[\varphi](\theta). \tag{17}$$

Note that when $\mathcal{M}$ is a 2D manifold (surface), $\Delta\mathcal{M}$ can also be called a discontinuity *curve*, as in Zhang et al. [2020]. We will often omit the dependency $[\varphi]$ or (θ) for simplicity.

The scalar field φ is continuous on each connected component of the continuous interior $\mathrm{Int}^{\circ}(\mathcal{M})$, so the integral over $\mathcal{M}$ can be represented as the sum of integrals over each connected component of $\mathrm{Int}^{\circ}(\mathcal{M})$. Then we can apply the transport theorem (14) for each connected component, and finally obtain the transport theorem with discontinuities:

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\mathcal{M}(\theta)} \varphi \mathrm{d}\mu_{\mathcal{M}} = \int_{\mathcal{M}} \left(\overset{\square}{\varphi} - \varphi\vec{\kappa}\cdot\vec{\mathcal{V}}_{\mathcal{M}}\right)\mathrm{d}\mu_{\mathcal{M}} + \int_{\partial\overline{\mathcal{M}}} \Delta\varphi \mathcal{V}_{\partial\overline{\mathcal{M}}}\mathrm{d}\mu_{\partial\overline{\mathcal{M}}}, \tag{18}$$

where $\Delta\varphi(\mathbf{x},\theta)$ on $\partial\overline{\mathcal{M}}[\varphi](\theta)$ is defined as:

$$\Delta\varphi(\mathbf{x},\theta) := \begin{cases} \varphi(\mathbf{x},\theta), & \text{if } \mathbf{x}\in\partial\mathcal{M} \\ \varphi^{-}(\mathbf{x},\theta) - \varphi^{+}(\mathbf{x},\theta), & \text{if } \mathbf{x}\in\Delta\mathcal{M}[\varphi](\theta) \end{cases}. \tag{19}$$

Here, $\varphi^{-}(\mathbf{x},\theta)$ and $\varphi^{+}(\mathbf{x},\theta)$ are defined as the limits of $\varphi(\mathbf{x},\theta)$ when approaching $\mathbf{x}$ from $-\mathbf{n}_{\Delta\mathcal{M}}(\mathbf{x},\theta)$ and $\mathbf{n}_{\Delta\mathcal{M}}(\mathbf{x},\theta)$, respectively. Note that unit normal vectors $\mathbf{n}_{\Delta\mathcal{M}}(\mathbf{x},\theta)$ and $\mathbf{n}_{\partial\overline{\mathcal{M}}}(\mathbf{x},\theta)$, the scalar normal velocities $\mathcal{V}_{\Delta\mathcal{M}}$ and $\mathcal{V}_{\partial\overline{\mathcal{M}}}$, and measures $\mu_{\Delta\mathcal{M}}$ and $\mu_{\partial\overline{\mathcal{M}}}$ can be defined in similar ways to $\partial\mathcal{M}$.

1.5 Generalization to Vector Parameters

Taking multiple parameters $\boldsymbol{\theta} = (\theta_1 ... \theta_d)$ into account is easily achievable by repeating the formulations for each parameter θ_i. Then $\mathbb{R}^n$ vector velocity terms v, $v_{\tan}$, and $\vec{\mathcal{V}}$ change to $\mathbb{R}^{n\times d}$ Jacobians, and scalar velocity and derivative terms $\mathcal{V}$, $\dot{\varphi}$, and $\overset{\square}{\varphi}$ change to $\mathbb{R}^{1\times d}$ gradients. Then changing the inner product term $\vec{\kappa}\cdot\vec{\mathcal{V}}_{\mathcal{M}}$ into a matrix product $\vec{\mathcal{V}}_{\mathcal{M}}^{T}\vec{\kappa}$ from Equations (14) and (18) generalizes the trasport theorem for multiple parameters. However, since the transport theorem for multiple parameters is equivalent to enunciating the theorem for each parameter, we keep writing formulations for a single parameter θ for the sake of simplicity.

2 DERIVATION DETAILS ON DIFFERENTIAL TRANSIENT PATH INTEGRAL

2.1 Formal Definition of Scene Geometry

In physically-based rendering, the scene geometry is usually represented as a 2D manifold, which is unfortunately riddled with discontinuities at the edges of the polygons. To clarify how to treat this discontinuity, we first define piece-wise differentiable manifolds in this section.

The *scene geometry* $\mathcal{M}$ is a piece-wise differentiable 2D manifold, which satisfies that:

- $\mathcal{M}$ is represented as a finite union of differentiable 2D manifold,s i.e., $\mathcal{M} = \bigcup_{i=1}^{p}\mathcal{M}_{[i]}$.
- For any two distinct pieces $\mathcal{M}_{[i]}$ and $\mathcal{M}_{[j]}$, if they intersect ($\mathcal{M}_{[i]}\cap\mathcal{M}_{[j]} \neq \phi$) then the intersection $\mathcal{M}_{[i]}\cap\mathcal{M}_{[j]}$ is a differentiable 1D manifold (curve).

Then the *boundary of the piece-wise differentiable 2D manifold* $\mathcal{M}$ is a piece-wise differentiable 1D manifold, defined as $\partial\mathcal{M} := \bigcup_{i=1}^{p}\partial\mathcal{M}_{[i]}$, and the *interior of the piece-wise 2D differentiable manifold* $\mathcal{M}$ is defined as $\mathrm{Int}(\mathcal{M}) := \mathcal{M} - \partial\mathcal{M} = \bigcup_{i=1}^{k}\mathrm{Int}\left(\mathcal{M}_{[i]}\right)$. Note that in usual polygonal representations, each differentiable piece $\mathcal{M}_{[i]}$ corresponds to each planar polygon (usually triangles), and the boundary of the scene geometry $\partial\mathcal{M}$ becomes the union of all edges in all polygon meshes. We do not consider self-intersection of scene geometry.

When a point $\mathbf{x}$ is in the boundary of the scene $\partial\mathcal{M}$, we observe that $\mathbf{x}$ belongs to one of two cases: it is either contained in exactly one differentiable piece $\partial\mathcal{M}_{[i]}$, or in the intersection of two differentiable pieces $\partial\mathcal{M}_{[i]}\cap\partial\mathcal{M}_{[j]}$. We will call the first case the *boundary edges*, and the second case the *sharp edges* [Zhang et al. 2020, 2019]. Then the boundary of the scene geometry $\partial\mathcal{M}$ becomes the union of the boundary edges and the sharp edges.

Generally, we can define piece-wise differentiable m-manifolds in a similar way, and extend the transport theorem in Equation (18) to piece-wise differentiable manifolds by taking the summation of the equation for each differentiable piece. Then we can use the same equation defining the transport theorem on a piece-wise differentiable $2(k+1)$-manifold $\Omega_k = \mathcal{M}^{k+1}$.

2.2 Product Space Rules

When an evolving manifold $\mathcal{N}(\theta)$ is formed as the product of two other evolving manifolds, $\mathcal{N}(\theta) = \mathcal{M}_1(\theta)\times\mathcal{M}_2(\theta)$, and there is a scalar function $\varphi(\cdot,\theta) : \mathcal{N}(\theta)\to\mathbb{R}$ which is the product of scalar functions $\varphi_1(\cdot,\theta) : \mathcal{M}_1(\theta)\to\mathbb{R}$ and $\varphi_2(\cdot,\theta) : \mathcal{M}_2(\theta)\to\mathbb{R}$, i.e., $\varphi(\mathbf{x}_1,\mathbf{x}_2,\theta) = \varphi_1(\mathbf{x}_1,\theta)\varphi(\mathbf{x}_2,\theta)$, we can use the same transport theorem described in Equation (18) by substituting $\mathcal{M}$ by $\mathcal{N}$. Then Equation (18) can be evaluated in terms of $\mathcal{M}_1$ and $\mathcal{M}_2$ as follows:

$$\begin{aligned}
\overset{\square}{\varphi}(\mathbf{x}_1,\mathbf{x}_2,\theta) &= \overset{\square}{\varphi}_1(\mathbf{x}_1,\theta)\varphi_2(\mathbf{x}_2,\theta) + \varphi_1(\mathbf{x}_1,\theta)\overset{\square}{\varphi}_2(\mathbf{x}_2,\theta) \\
\vec{\kappa}_{\mathcal{N}}(\mathbf{x}_1,\mathbf{x}_2,\theta) &= \left(\vec{\kappa}_{\mathcal{M}_1}(\mathbf{x}_1,\theta), \vec{\kappa}_{\mathcal{M}_2}(\mathbf{x}_2,\theta)\right), \\
\vec{\mathcal{V}}_{\mathcal{N}}(\mathbf{x}_1,\mathbf{x}_2,\theta) &= \left(\vec{\mathcal{V}}_{\mathcal{M}_1}(\mathbf{x}_1,\theta), \vec{\mathcal{V}}_{\mathcal{M}_2}(\mathbf{x}_2,\theta)\right), \\
\partial\overline{\mathcal{N}}[\varphi](\theta) &= \partial\overline{\mathcal{M}_1}[\varphi_1](\theta)\times\mathcal{M}_2(\theta)\cup\mathcal{M}_1(\theta)\times\partial\overline{\mathcal{M}_2}[\varphi_2](\theta), \\
\Delta\varphi(\mathbf{x}_1,\mathbf{x}_2,\theta) &= \begin{cases} \varphi_2\Delta\varphi_1 & \text{if } (\mathbf{x}_1,\mathbf{x}_2)\in\partial\overline{\mathcal{M}_1}\times\mathcal{M}_2 \\ \varphi_1\Delta\varphi_2 & \text{if } (\mathbf{x}_1,\mathbf{x}_2)\in\mathcal{M}_1\times\partial\overline{\mathcal{M}_2} \end{cases}, \\
\mathcal{V}_{\partial\overline{\mathcal{N}}}(\mathbf{x}_1,\mathbf{x}_2,\theta) &= \begin{cases} \varphi_2\mathcal{V}_{\partial\overline{\mathcal{M}_1}} & \text{if } (\mathbf{x}_1,\mathbf{x}_2)\in\partial\overline{\mathcal{M}_1}\times\mathcal{M}_2 \\ \varphi_1\mathcal{V}_{\partial\overline{\mathcal{M}_2}} & \text{if } (\mathbf{x}_1,\mathbf{x}_2)\in\mathcal{M}_1\times\partial\overline{\mathcal{M}_2} \end{cases}.
\end{aligned} \tag{20}$$

These rules can be extend in a similar way to an arbitrary number of products, such as the order-k path space $\Omega_k = \mathcal{M}^{k+1}$.

2.3 Terms in the Path Integral

Recall the transient path integral, the path throughput, and the correlated importance described in the main paper, respectively, for negligible scattering delays surface path vertices:

$$I = \int_{\Omega} f_{\mathcal{T}}(\bar{\mathbf{x}})\,\mathrm{d}\mu(\bar{\mathbf{x}}), \tag{21}$$

$$f_{\mathcal{T}}(\bar{\mathbf{x}}) := \mathfrak{T}(\bar{\mathbf{x}})\,S_e(\bar{\mathbf{x}}), \tag{22}$$

$$\mathfrak{T}(\bar{\mathbf{x}}) := \left[\prod_{i=1}^{k-1}\rho(\mathbf{x}_{i-1},\mathbf{x}_i,\mathbf{x}_{i+1})\right]\left[\prod_{i=0}^{k-1}G(\mathbf{x}_i,\mathbf{x}_{i+1})\,V(\mathbf{x}_i,\mathbf{x}_{i+1})\right], \tag{23}$$

$$S_e(\bar{\mathbf{x}}) = \int_{-\infty}^{\infty} L_e(\mathbf{x}_0,\mathbf{x}_1,t)\,W_e(\mathbf{x}_{k-1},\mathbf{x}_k,t+\mathrm{tof}(\bar{\mathbf{x}}))\,\mathrm{d}t. \tag{24}$$

For a mathematically rigorous derivation, we define each term in the path integral: L_e, ρ, W_e, G, and V.

Definition 2.1. *For given scene geometry $\mathcal{M}$, the geometric function $G\colon \mathrm{Int}(\mathcal{M}) \times \mathrm{Int}(\mathcal{M}) \to \mathbb{R}$ is defined as:*

$$G(\mathbf{x},\mathbf{y}) := \begin{cases} \frac{|\hat{n}_{\mathbf{x}}\cdot\hat{\omega}_{\mathbf{xy}}||\hat{n}_{\mathbf{y}}\cdot\hat{\omega}_{\mathbf{yx}}|}{\|\mathbf{x}-\mathbf{y}\|^2} & \mathbf{x}\neq\mathbf{y} \\ 0 & \mathbf{x}=\mathbf{y} \end{cases}. \tag{25}$$

Also, when the domain is restricted to differentiable pieces of the scene geometry, denoted by $\mathrm{Int}\left(\mathcal{M}_{[i]}\right) \times \mathrm{Int}\left(\mathcal{M}_{[j]}\right)$ $(1 \le i, j \le p)$, then the restricted function can be continuously extended onto $\mathcal{M}_{[i]} \times \mathcal{M}_{[j]}$, which contains their boundaries $\partial\mathcal{M}_{[i]}$ and $\partial\mathcal{M}_{[j]}$. We will denote this function as $G|_{\mathcal{M}_{[i]}\times\mathcal{M}_{[j]}} : \mathcal{M}_{[i]} \times \mathcal{M}_{[j]} \to \mathbb{R}$, where $G|_{\mathcal{M}_{[i]}\times\mathcal{M}_{[j]}}$ is a continuous function. Note that the entire geometric function G satisfies $\Delta\left(\mathcal{M}^2\right)[G] = \partial\left(\mathcal{M}^2\right)$.

Definition 2.2. *For given scene geometry $\mathcal{M}$, the visibility function $V\colon \mathcal{M}\times\mathcal{M} \to \mathbb{R}$ is defined as:*

$$V(\mathbf{x},\mathbf{y}) := \begin{cases} 1 & \mathrm{openlineseg}(\mathbf{x},\mathbf{y}) \cap \mathcal{M} = \phi \\ 0 & otherwise \end{cases}, \tag{26}$$

where $\mathrm{openlineseg} := \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \mid 0 < \lambda < 1\}$ means the open line segment between two given points. Unlike the geometric function G and the visibility function V, the light source emission function L_e, surface scattering function (BSDF) ρ, and the sensor sensitivity function W_e vary depending on the scene. We can introduce the following conditions which those terms should satisfy in practice:

- A.1 For any fixed $\mathbf{x}_0, \mathbf{x}_1 \in \mathcal{M}$, $L_e(\mathbf{x}_0 \to \mathbf{x}_1, \cdot) : \mathbb{R} \to \mathbb{R}$ as a function of t contains a finite number of jump discontinuities at $t_{L1}...t_{Lq}$ and a finite number of Dirac delta distributions at $t_1...t_r$. Except for $t_{L1}...t_{Lq}$ and $t_1...t_r$, $L_e(\mathbf{x}_0 \to \mathbf{x}_1, \cdot)$ is continuous. Also, $t_{L1}...t_{Lq}$ and $t_1...t_r$ vary continuously when $\mathbf{x}_0$ and $\mathbf{x}_1$ vary continuously.
- A.2 For any fixed $\mathbf{x}_{N-1}, \mathbf{x}_N \in \mathcal{M}$, $W_e(\mathbf{x}_{N-1} \to \mathbf{x}_N, \cdot) : \mathbb{R} \to \mathbb{R}$ as a function of t contains a finite number of jump discontinuities at $t_{W1}...t_{Ws}$. Except for $t_{W1}...t_{Ws}$, $W_e(\mathbf{x}_0 \to \mathbf{x}_1, \cdot)$ is continuous. Also, $t_{W1}...t_{Ws}$ vary continuously when $\mathbf{x}_0$ and $\mathbf{x}_1$ vary continuously.
- A.3 The source emission L_e does not have non-zero energy on spatially zero-measure sets.
- A.4 The scattering function ρ does not contain a Dirac delta, i.e., there is no ideal specular reflection, and is continuous except when the incoming or outgoing directions are perpendicular to the surface normal.

A.1 and A.2 are our novel assumptions for transient rendering, while A.3 and A.4 are common assumptions used in physically-based differentiable rendering [Bangaru et al. 2020; Li et al. 2018; Loubet et al. 2019; Zhang et al. 2020, 2019]. Note that these assumptions cover most of practical cases.

2.4 Differential Transient Path Integral

To differentiate Equation (21) using the transport theorem in Equation (18), we will first evaluate the boundary path space $\partial\overline{\Omega}[f_{\mathcal{T}}](\theta)$. For simplicity, we first fix k and evaluate the order-k boundary path space $\partial\overline{\Omega_k}$. By definition of extended boundary (Equation (17)), $\partial\overline{\Omega_k}$ consists of the (geometric) boundary $\partial\Omega_k$ and the discontinuity set $\Delta\Omega_k[f_{\mathcal{T}}]$. Since the total throughput $f_{\mathcal{T}}$ is the product of partial terms S_e, ρ, V, and G, the entire discontinuity set can be represented as the union of discontinuity sets caused by each of those partial terms as follows:

$$\begin{aligned} \partial\overline{\Omega_k}[f_{\mathcal{T}}] =& \partial\Omega_k \cup \Delta\Omega_k[f_{\mathcal{T}}] \\ =& \partial\Omega_k \cup \Delta\Omega_k[S_e] \cup \Delta\Omega_k[G_1\cdots G_k] \\ & \cup \Delta\Omega_k[V_1\cdots V_k] \cup \Delta\Omega_k[\rho_1\cdots\rho_{k-1}]. \end{aligned} \tag{27}$$

Note that both boundary edges and sharp edges are contained in $\partial\Omega_k$ as mentioned in §2.1.

Applying the product space rule Eq. (20) to Eq. (27), $\Delta\Omega_k[G_1\cdots G_k]$, $\Delta\Omega_k[V_1\cdots V_k]$, and $\Delta\Omega_k[\rho_1\cdots\rho_{k-1}]$ can be rewritten as follows:

$$\begin{aligned} \Delta\Omega_k[G_1\cdots G_k] &= \bigcup_{i=1}^{k} \mathcal{M}_0 \times \cdots \mathcal{M}_{i-2} \times \Delta\mathcal{M}^2[G] \times \mathcal{M}_{i+1} \times \cdots \mathcal{M}_k, \\ \Delta\Omega_k[V_1\cdots V_k] &= \bigcup_{i=1}^{k} \mathcal{M}_0 \times \cdots \mathcal{M}_{i-2} \times \Delta\mathcal{M}^2[V] \times \mathcal{M}_{i+1} \times \cdots \mathcal{M}_k, \\ \Delta\Omega_k[\rho_1\cdots\rho_{k-1}] &= \bigcup_{i=1}^{k-1} \mathcal{M}_0 \times \cdots \mathcal{M}_{i-2} \times \Delta\mathcal{M}^3[\rho] \times \mathcal{M}_{i+2} \times \cdots \mathcal{M}_k. \end{aligned} \tag{28}$$

Vanishing and overlapping discontinuities. Some discontinuity sets of partial terms, $\Delta\Omega_k[S_e]$, $\Delta\mathcal{M}^2[G]$, $\Delta\mathcal{M}^2[V]$, and $\Delta\mathcal{M}^3[\rho]$ may not contribute to the total discontinuity set. As discussed in previous work [Zhang et al. 2020], $\Delta\mathcal{M}^3[\rho]$ is caused at the horizontal ray direction, but in that case the geometric term G is continuously zero. For a formal mathematical analysis we rely on the following proposition:

Proposition 2.3. *Given a set $X \in \mathbb{R}^n$ and a subset $A \subset X$, suppose that a function $\varphi_1\colon X \to \mathbb{R}$ is continuously zero on A and a function $\varphi_2\colon X \to \mathbb{R}$ is a bounded function. Then even if φ_2 is discontinuous in A, A does not contribute to the discontinuity set of $\varphi_1\varphi_2$, i.e., $A \cap \Delta X[\varphi_1\varphi_2] = \phi$.*

Thus, we can omit $\Delta\Omega_k[\rho_1\cdots\rho_{k-1}]$ from Equation (27). The visibility function V also presents a discontinuity at the horizontal ray direction, which vanished by the geometric function G. Then Equation (27) can be rewritten in the following simpler form:

$$\begin{aligned} \partial\overline{\Omega}_k[f_{\mathcal{T}}] =& \partial\Omega_k \cup \Delta\Omega_k[S_e] \\ & \cup (\Delta\Omega_k[G_1V_1\cdots G_kV_k] - \partial\Omega_k). \end{aligned} \tag{29}$$

Note that in the last term of Equation (29) we will evaluate the discontinuity set of the product term GV directly, without considering discontinuities on $\partial\Omega_k$, already accounted for in the first term of Equation (29).

2.4.1 Discontinuity of GV terms. The term $(\Delta\Omega_k[G_1V_1\cdots G_kV_k] - \partial\Omega_k)$ caused by silhouette edges can be represented by the product rule (Equation (20)) as:

$$\begin{aligned} \Delta\Omega_k[G_1V_1\cdots G_kV_k] - \partial\Omega_k = \bigcup_{i=1}^{k} \mathcal{M}_0 \times \cdots \mathcal{M}_{i-2} \\ \times \left(\Delta\mathcal{M}^2[GV] - \partial\mathcal{M}^2\right) \times \mathcal{M}_{i+1} \times \cdots \mathcal{M}_k, \end{aligned} \tag{30}$$

where the term $\left(\Delta\mathcal{M}^2[GV] - \partial\mathcal{M}^2\right)$ is obtained by the following lemma:

Lemma 2.4. *The discontinuity submanifold of GV can be specified as:*

$$\begin{aligned}\Delta\mathcal{M}^2[GV] - \partial\mathcal{M}^2 =& \Delta\left(\mathrm{Int}\left(\mathcal{M}^2\right)\right)[GV] \\ =& \{(\mathbf{x},\mathbf{y}) \in \mathrm{Int}\left(\mathcal{M}^2\right) \mid \text{line segment between } \mathbf{x},\mathbf{y} \\ & \text{only intersects silhouette edges of } \mathcal{M}\}.\end{aligned} \tag{31}$$

A line segment between $(\mathbf{x},\mathbf{y})$ intersects a silhouette edge of $\mathcal{M}$ if and only if:

$$\begin{aligned}&\exists \mathbf{z} \in \mathrm{openlineseg}(\mathbf{x},\mathbf{y}) \cap \mathcal{M} \text{ s.t. one of the three cases holds:} \\ &\quad \mathbf{z} \in \mathrm{Int}(\mathcal{M}) \text{ and } \mathbf{n}(\mathbf{z}) \cdot (\mathbf{y}-\mathbf{x}) = 0, \\ &\quad \mathbf{z} \in \partial\mathcal{M}_{[i]} \text{ (boundary edge), or} \\ &\quad [\mathbf{z} \in \partial\mathcal{M}_{[i]} \cap \partial\mathcal{M}_{[j]} \text{ with } [i] \neq j \text{ (sharp edge) and} \\ &\quad (\mathbf{n}_i(\mathbf{z}) \cdot \mathbf{y} - \mathbf{x})\left(\mathbf{n}_j(\mathbf{z}) \cdot (\mathbf{y}-\mathbf{x})\right) \leq 0].\end{aligned} \tag{32}$$

To prove the lemma, we first show that $V^{-1}(\{1\}) \cap \mathrm{Int}\left(\mathcal{M}^2\right) \subset \mathrm{Int}^\circ\left(\mathcal{M}^2\right)[GV]$, i.e. for any pair of mutually visible points $(\mathbf{x},\mathbf{y}) \in \mathrm{Int}\left(\mathcal{M}^2\right)$ with $V(\mathbf{x},\mathbf{y}) = 1$, GV is continuous on $(\mathbf{x},\mathbf{y})$. If $\mathbf{y}-\mathbf{x}$ is orthogonal to $\mathbf{n}(\mathbf{x})$ or $\mathbf{n}(\mathbf{y})$, the claim holds since G is continuously zero on $(\mathbf{x},\mathbf{y})$. Otherwise, we can easily observe that the visible pair of points are still visible when they move within a sufficiently small distance.

Second, we show that if $(\mathbf{x},\mathbf{y}) \in V^{-1}(\{0\}) \cap \mathrm{Int}\left(\mathcal{M}^2\right)$ and the line segment between them intersects a point $z \in \mathcal{M}$ which is not on a silhouette edge, then V is continuous on $(\mathbf{x},\mathbf{y})$. Note that by definition of silhouette described in Lemma 2.4, $\mathbf{z}$ belongs to one of two cases: i) $\mathbf{z} \in \mathrm{Int}(\mathcal{M})$ and $\mathbf{n}(\mathbf{z}) \cdot (\mathbf{y}-\mathbf{x})$, or ii) $\mathbf{z} \in \partial\mathcal{M}_{[i]} \cap \partial\mathcal{M}_{[j]}$ with $i \neq j$ (sharp edge) and $(\mathbf{n}_i(\mathbf{z}) \cdot \mathbf{y} - \mathbf{x})\left(\mathbf{n}_j(\mathbf{z}) \cdot (\mathbf{y}-\mathbf{x})\right) > 0$. In both cases, the line segment penetrates the open disk which neighbors $\mathbf{z}$. Then we observe that when $\mathbf{x}$ and $\mathbf{y}$ moves within a sufficiently small distance, the line segment between them still penetrates the open disk.

2.4.2 Discontinuities of the S_e term. Recall the assumptions A.1 and A.2 for source emission L_e and sensor sensitivity W_e. Given $\mathbf{x}_0$ and $\mathbf{x}_1$, $L_e(\mathbf{x}_0 \longrightarrow \mathbf{x}_1, \cdot)$ has Dirac delta mass at $t = t_{L1} \ldots t_{Lj}$, and given $\mathbf{x}_{k-1}$ and $\mathbf{x}_k$, $W_e(\mathbf{x}_{k-1} \longrightarrow \mathbf{x}_k, \cdot)$ has discontinuity at $t = t_{W1} \ldots t_{Wk}$. By the following lemma, we can obtain the discontinuity of S_e.

Lemma 2.5. *With assumptions A.1 and A.2, the discontinuity of S_e can be determined as:*

$$\begin{aligned}\Delta\Omega_k[S_e] = \{\bar{\mathbf{x}} \in \Omega_k \mid & \mathrm{tof}(\bar{\mathbf{x}}) = t_{Wi} - t_j \\ & \text{for some } 1 \leq i \leq s,\ 1 \leq j \leq r\}.\end{aligned} \tag{33}$$

Note that this representation does not include a vanishing part for $G = 0$. To prove this lemma, first let $u_{t_0}(t)$ denote the unit step function, which is defined as $u_{t_0}(t) = 1$ for $t > t_0$ and $u_{t_0}(t) = 0$ otherwise. Also, $\delta_{t_0}(t)$ denotes the Dirac delta function centered at t_0 so that $\delta_{t_0}(t) = \delta(t - t_0)$. Important facts to prove Lemma 2.5 is that $u_{t_0} \star u_{t_1}$ is a continuous function and $\delta_{t_0} \star u_{t_1} = u_{t_1 - t_0}$, where $\star$ denotes the cross-correlation. Since $S_e(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_{k-1}, \mathbf{x}_k, t = \mathrm{tof}(\bar{\mathbf{x}}))$ is the correlation of L_e and W_e in the temporal domain, the only discontinuity occurs when tof $(\bar{\mathbf{x}})$ lies on the discontinuous point of $\delta_{t_j} \star u_{t_{Wi}}$.

For example, suppose that the source is $L_e(\mathbf{x}_0, \mathbf{x}_1, t) = L_e(\mathbf{x}_0, \mathbf{x}_1)\,\delta(t)$ and the sensor is $W_e(\mathbf{x}_{N-1}, \mathbf{x}_N, t) = W_e(\mathbf{x}_{k-1}, \mathbf{x}_k)\,\mathrm{box}(t; t_{\mathrm{start}}, t_{\mathrm{end}})$ where $\mathrm{box}(t; t_{\mathrm{start}}, t_{\mathrm{end}})$ refers to the unit box function starting from t_{start} and ending at t_{end}. In this case the discontinuity of S_e, $\Delta\Omega_k[S_e]$, is the set of paths with travel time t_{start} or t_{end}. In practice, if the light source and the sensor are not Dirac deltas then $\Delta\Omega_k[S_e] = \phi$.

Boundary Contribution $\Delta f_{\mathcal{T}}$ and Normal Velocities $\mathcal{V}_{\partial\overline{\Omega}_k}$. When $\Delta\Omega_k[S_e]$ becomes an empty set, the boundary path space $\partial\overline{\Omega}_k$ consists of $\partial\Omega_k$ and $\Delta\Omega_k[G_1V_1 \cdots G_kV_k] - \partial\Omega_k$. For a global parameterization we only need to concern ourselves with the discountinuity of the GV term $\Delta\Omega_k[G_1V_1 \cdots G_kV_k] - \partial\Omega_k$. Then the discontinuity comes from the visibility, so $\Delta f_{\mathcal{T}}$ becomes the same as $f_{\mathcal{T}}$ and the normal velocity $\mathcal{V}_{\partial\overline{\Omega}_k}$ converges to its steady-state counterpart.

3 VALIDATION

We validate our results against the baseline transient renderer by Jarabo and colleagues [2014], using a short light pulse of 0.01 fs.

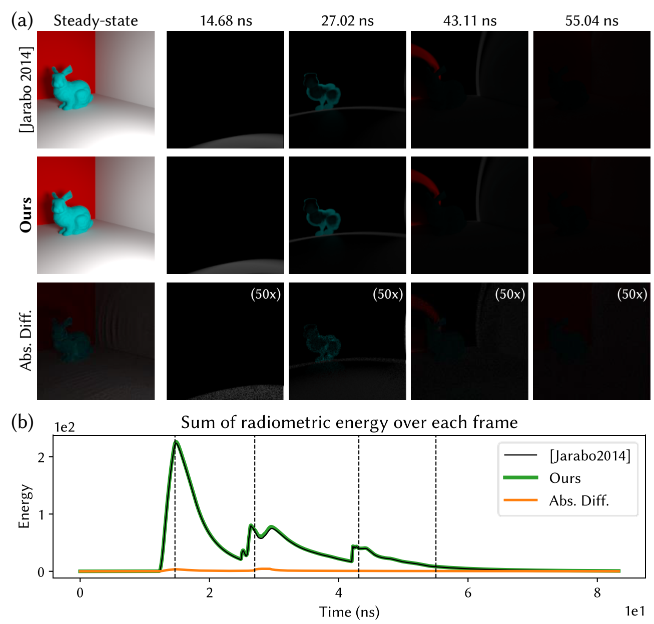

Fig. 1. Validation of our differentiable transient rendering against the transient path integral method of Jarabo et al. [2014]. (a) The transient sequences show how the pulse of light illuminates the floor first at 14.68 ns, then the bunny at 27.02 ns, and finally the red wall at 43.11 ns. At 55.04 ns there is almost no signal. The last row shows the absolute difference between the frames, amplified by a factor of 50. (b) Radiance and absolute difference plot. Dotted vertical lines indicate the exact frames shown above. Our framework matches the baseline closely.

4 ALGORITHM TABLES

We append Algorithms 1 and 2 for estimating the interior and boundary integrals, respectively. Variables d and $\dot{d}$ store the optical path

ALGORITHM 1: Estimating the interior integral

Data: scene, pixel index (i, j), max bounce k
Result: Rendered temporal histogram of the (i, j)-th pixel $I[i, j, \cdot]$ and its scene derivative $\dot{I}[i, j, \cdot]$

$\mathbf{x}[k+2], \mathbf{y}[k+1] \leftarrow$ New arrays of 3D positions on the scene geometry;
$d_{\mathbf{x}}[k+2], \dot{d}_{\mathbf{x}}[k+2] \leftarrow$ New arrays of floating numbers (path distance);
$d_{\mathbf{y}}[k+1], \dot{d}_{\mathbf{y}}[k+1] \leftarrow$ New arrays of floating numbers (path distance);
$f_{\mathbf{x}}[k+2], \dot{f}_{\mathbf{x}}[k+2] \leftarrow$ New arrays of floating numbers (throughput);
$f_{\mathbf{y}}[k+1], \dot{f}_{\mathbf{y}}[k+1] \leftarrow$ New arrays of floating numbers (throughput);
$\mathbf{x}[0] \leftarrow$ camera position, $\left(f_{\mathbf{x}}[0], \dot{f}_{\mathbf{x}}[0]\right) \leftarrow (1, 0)$;
for $1 \le i < k+2$ **do**
  if $i = 1$ **then**
    Sample $(\hat{\omega}_o, p) \sim \mathbb{P}_{\text{cameraPrimiryRay},ij}$;
  else
    Sample $(\hat{\omega}_o, p) \sim \mathbb{P}_{\text{brdf}}(\mathbf{x}[i-1], \hat{\omega}_i, \cdot)$;
  $\mathbf{x}_{\text{temp}} \leftarrow \text{rayTrace}(\mathbf{x}[i-1], \hat{\omega}_o)$;
  if $\mathbf{x}_{temp}$ *is valid* **then**
    $\mathbf{x}[i] \leftarrow \mathbf{x}_{\text{temp}}$;
    $(\alpha, \dot{\alpha}) \leftarrow$ The value and scene derivative of:
     $\rho(\mathbf{x}[i-2] \to \mathbf{x}[i-1] \to \mathbf{x}[i])\, G(\mathbf{x}[i-1], \mathbf{x}[i])\, J(\mathbf{x}[i])$;
    $p \leftarrow p\, |\hat{n}(\mathbf{x}[i]) \cdot -\hat{\omega}_o| / \|\mathbf{x}[i] - \mathbf{x}[i-1]\|^2$;
    $f_{\mathbf{x}}[i] \leftarrow f_{\mathbf{x}}[i-1]\, \alpha / p$;
    $\dot{f}_{\mathbf{x}}[i] \leftarrow \left(\dot{f}_{\mathbf{x}}[i-1]\, \alpha + f_{\mathbf{x}}[i-1]\, \dot{\alpha}\right) / p$;
    $\left(\delta, \dot{\delta}\right) \leftarrow$ The value and scene derivative of:
     $\eta\, \|\mathbf{x}[i] - \mathbf{x}[i-1]\|$;
    $d_{\mathbf{x}}[i] \leftarrow d_{\mathbf{x}}[i-1] + \delta$;
    $\dot{d}_{\mathbf{x}}[i] \leftarrow \dot{d}_{\mathbf{x}}[i-1] + \dot{\delta}$;
    $\hat{\omega}_i \leftarrow -\hat{\omega}_o$;
  else
    break;
Sample $\mathbf{y}[0] \sim \mathbb{P}_{\text{emitter}}$, $f_{\mathbf{y}}[0] \leftarrow 1/\mathbb{P}_{\text{emitter}}(\mathbf{y}[0])$;
Similarly construct the light subpath $\mathbf{y}[\,], d_{\mathbf{y}}[\,], \dot{d}_{\mathbf{y}}[\,], f_{\mathbf{y}}[\,], \dot{f}_{\mathbf{y}}[\,]$.;
for $0 \le s \le k-1$ **do**
  $\left(f, \dot{f}\right) \leftarrow \text{combineSubpaths}(\mathbf{x}[0:s+1], \mathbf{y}[0:k-s])$;
  $\left(\delta, \dot{\delta}\right) \leftarrow$ The value and scene derivative of:
   $\eta\, \|\mathbf{x}[s] - \mathbf{y}[k-s-1]\|$;
  $d \leftarrow d_{\mathbf{x}}[s] + d_{\mathbf{y}}[k-s-1] + \delta$;
  $\dot{d} \leftarrow \dot{d}_{\mathbf{x}}[s] + \dot{d}_{\mathbf{y}}[k-s-1] + \dot{\delta}$;
  $w \leftarrow$ CombinationStrategy; // Use Chapter 9 in [Veach 1997]
  for $l \in S_e.\text{range}(d)$ **do**
    $s \leftarrow S_e[l](\mathbf{y}[0], \mathbf{y}[1], \mathbf{x}[1], \mathbf{x}[0], d/c)$;
    $\dot{s} \leftarrow \dot{S_e}[l](\mathbf{y}[0], \mathbf{y}[1], \mathbf{x}[1], \mathbf{x}[0], d/c)$;
    $I[i, j, l] \leftarrow I[i, j, l] + w f s$;
    $\dot{I}[i, j, l] \leftarrow \dot{I}[i, j, l] + w \dot{f} s + w f \dot{s}$;

ALGORITHM 2: Estimating the boundary integral

Data: scene, pixel index (i, j), max bounce k
Result: Rendered scene derivative temporal histogram of the (i, j)-th pixel $\dot{I}[i, j, \cdot]$

Sample $(\mathbf{x}_B, \hat{\omega}_B) \leftarrow \mathbb{P}_{\text{boundaryRay}}$;
$\mathbf{x}_L \leftarrow \text{rayTrace}(\mathbf{x}_B, -\hat{\omega}_B)$;
$\mathbf{x}_S \leftarrow \text{rayTrace}(\mathbf{x}_B, \hat{\omega}_B)$;
if $\mathbf{x}_L$ *and* $\mathbf{x}_S$ *are valid* **then**
  $f_B \leftarrow$
   $G(\mathbf{x}_L, \mathbf{x}_S)\, \mathcal{V}_{\partial\hat{\Omega}}(\mathbf{x}_L, \mathbf{x}_S)\, J_B(\mathbf{x}_B, \hat{\omega}_B) / \mathbb{P}_{\text{boundaryRay}}(\mathbf{x}_B, \hat{\omega}_B)$;
  $d_B \leftarrow \eta\, \|\mathbf{x}_L - \mathbf{x}_S\|$;
  $(f_S, d_S, \mathbf{x}_0, \mathbf{x}_1) \leftarrow \text{EstimateSensorSubpath}(\mathbf{x}_S)$;
  $(f_L, d_L, \mathbf{y}_0, \mathbf{y}_1) \leftarrow \text{EstimateSourceSubpath}(\mathbf{x}_L)$;
  $d \leftarrow d_L + d_B + d_S$;
  for $l \in S_e.\text{range}(d)$ **do**
    $s \leftarrow S_e[l](\mathbf{y}_0, \mathbf{y}_1, \mathbf{x}_1, \mathbf{x}_0, d/c)$;
    $\dot{I}[i, j, l] \leftarrow \dot{I}[i, j, l] + f_L f_B f_S s$;

length and its derivative with respect to the scene parameter, respectively. In both algorithms, $S_e.\text{range}(d)$ returns the list of indices of temporal bins in which the sampled path is stored. In Algorithm 2, EstimateSensorSubpath and EstimateSourceSubpath can be performed as in Algorithm 1, without computing the derivative terms $\dot{f}$ and $\dot{d}$.

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