Supplemental Document: Spin-Weighted Spherical Harmonics for Polarized Light Transport

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PREFACE

This supplemental document serves several purposes for different readers, with the exception of Sections 1.1 and 1.2, which are recommended for all readers. First, Sections 2.1 to 2.3, 2.6, 3.1, 4.2, 4.1 to 4.4 provide some additional motivation and detail to the background described in Sections 4 and 5 of the main paper for readers who are not familiar with either spherical harmonics or polarization. Second, the remainder of this document contains formal definitions and detailed steps for proofs in a more axiomatic and rigorous manner. This remainder is intended for more dedicated readers who want to verify the mathematical properties of polarized spherical harmonics presented in the main paper. Since each of Sections 2, 3, and 4 contains subsections intended for different readers, we also clarify the purpose of each subsection at the beginning of each of these sections.

Since our work deals with extensions of quantities and equations that have been previously treated in spherical harmonics and polarization, it is helpful to see Table 1, which compares the formulae proposed in this work with the existing formulae to which each corresponds.

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Contents

1 PRELIMINARIES

1.1 Geometric and Numeric Quantities

In this paper, we investigate various categories of quantities such as vectors, Stokes vectors, transforms, and functions on the unit sphere to these quantities. Before discussing individual concepts of them, we first distinguish them into two categories, *geometric* quantities and *numeric* quantities, inspired by a computer graphics textbook [Gortler 2012].

Geometric quantities can be easily understood as physical quantities, which we can see in the real world, and numeric quantities can be considered as just arrays of numbers. For example, we call *vectors* (or *geometric vectors* to clearly avoid confusion of terminology), denoted by $\vec{a}, \vec{v}, \hat{\omega}, \cdots \in \vec{\mathbb{R}}^3$, as geometric quantities, and *numeric vectors*, denoted by $\mathbf{a}, \mathbf{c}, \mathbf{x} \in \mathbb{R}^n$, as numeric quantities. While vectors discussed in this paper are always three-dimensional quantities, numeric vectors include four-dimensional Stokes component vectors, which is discussed in Section 4.2 in the main paper, and spherical harmonics coefficient vectors with arbitrary dimension, which is discussed in Section 4.1 in the main paper. We will call the numerical representation of vectors as *coordinate vectors*, which are special cases of numeric vectors.

Regardless of whether geometric or numeric, we call a set with well-defined addition and scalar multiplication a *linear space*[1], in the sense of linear algebra. Both the set of (geometric) vectors and the set of numeric vectors are linear spaces.

For sets X and Y, $\mathcal{F}(X, Y) = \{f : X \to Y\}$ [2] denotes the set of all functions from X into Y. If X and Y are linear spaces, $\mathcal{L}(X, Y) = \{f \in \mathcal{F}(X, Y) \mid f(ax + by) = af(x) + bf(y)\}$ indicates the set of *linear maps* from X into Y, regardless whether geometric or numeric. We call linear maps between numeric vectors *matrices* and those between geometric vectors *transforms*. Moreover, a *frame* indicates an orthonormal[3] linear map from coordinate vectors to geometric vectors, and the set of frames is denoted by $\vec{\mathbb{F}}^3 := \left\{\vec{\mathbf{F}} \in \mathcal{L}\left(\vec{\mathbb{R}}^3, \mathbb{R}^3\right) \mid \vec{\mathbf{F}} \text{ is orthonormal}\right\}$. Then, we observe that a vector is equal to the matrix product of a frame and a coordinate vector as described in Figure 1(a). Note that a coordinate vector itself does not have any physical meaning in the real world, but it can be converted into a geometric vector by combining it with a frame.

Similar to the multiplication of frames and coordinate vectors, we have several kinds of multiplications as follows:

$$
\begin{aligned}
\text{matrix } (\in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)) \times \text{coordinate vector } (\in \mathbb{R}^3) &= \text{coordinate vector } (\in \mathbb{R}^3) \\
\text{frame } (\in \mathcal{L}(\mathbb{R}^3, \vec{\mathbb{R}}^3)) \times \text{coordinate vector } (\in \mathbb{R}^3) &= \text{vector } (\in \vec{\mathbb{R}}^3) \\
\text{transform } (\in \mathcal{L}(\vec{\mathbb{R}}^3, \vec{\mathbb{R}}^3)) \times \text{vector } (\in \vec{\mathbb{R}}^3)) &= \text{vector } (\in \vec{\mathbb{R}}^3) \\
\text{matrix } (\in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)) \times \text{matrix } (\in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)) &= \text{matrix } (\in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)) \\
\text{frame } (\in \mathcal{L}(\mathbb{R}^3, \vec{\mathbb{R}}^3)) \times \text{matrix } (\in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^3)) &= \text{frame } (\in \mathcal{L}(\mathbb{R}^3, \vec{\mathbb{R}}^3)) \\
\text{transform } (\in \mathcal{L}(\vec{\mathbb{R}}^3, \vec{\mathbb{R}}^3)) \times \text{frame } (\in \mathcal{L}(\mathbb{R}^3, \vec{\mathbb{R}}^3)) &= \text{frame } (\in \mathcal{L}(\mathbb{R}^3, \vec{\mathbb{R}}^3)) \\
\text{transform } (\in \mathcal{L}(\vec{\mathbb{R}}^3, \vec{\mathbb{R}}^3)) \times \text{transform } (\in \mathcal{L}(\vec{\mathbb{R}}^3, \vec{\mathbb{R}}^3)) &= \text{transform } (\in \mathcal{L}(\vec{\mathbb{R}}^3, \vec{\mathbb{R}}^3)).
\end{aligned} \tag{1}
$$

[1]This is more frequently called *vector space* in other literature, but we do not use it since the word 'vector' might be misunderstood as a geometric quantity.

[2]To consider it as an inner product space in later sections, $\mathcal{F}$ should contain additional conditions such as L2 integrability for mathematical rigor. For the sake of simplicity, however, we have omitted such conditions as they are always satisfied in practical cases. We refer to Groemer [1996] for complete mathematical rigor of the theory of spherical harmonics

[3]It may not be orthonormal in general, but we only consider orthonormal frames in this work for simplicity.

Table 1. Summary of our fundamental building blocks for frequency domain theory of polarized rendering, compared with conventional scalar rendering. All the quantities (3rd-7th rows) and operations (8th-10th rows) have their frequency domain formulations. l is the maximum level of spherical harmonics basis. Each cell contains references both for the main paper (denoted by "M-") and this supplemental document (denoted by "S-").

	Scalar rendering			**Polarized** rendering		
	Angular domain	Freq. domain (M-Sec.4.1, S-Sec.2)	# coeff.	Angular domain (M-Sec.4.2, S-Sec.3)	Freq. domain (M-Sec.6, S-Sec.5)	# coeff.
Radiance	Scalar radiance	–	–	Stokes vector	-	–
Environment map	spherical function (scalar field)	SH coeff. vector (M-Eq. (3), S-Eq. (35))	$O(l^2)$	Stokes vector field (M-Sec.5, S-Sec.4.1)	PSH coeff. vector (M-Eq. (30))	$O(4l^2)$
BRDF	BRDF	SH coeff. matrix (M-Eq. (4), S-Eq. (41))	$O(l^4)$	Mueller pBRDF	PSH coeff. matrix (M-Eq. (40))	$O(4 \times 4l^4)$
+ Axial symmetry	isotropic BRDF	+ Sparsity (M-Eq. (7), S-Prop.2.9)	$O(l^3)$	isotropic pBRDF	+ Sparsity (M-Eq. (52))	$O(4 \times 4l^3)$
+ Radial symmetry	convolution kernel (S-Prop.2.12)	+ Sparsity (M-Eq. (11))	$O(l)$	polarized convolution kernel (M-Eq. (62), S-Prop. 5.13)	+ Sparsity (M-Eq. (68), S-Eqs. (158,164,175,176))	$O(4 \times 4l)$
Rotation	(M-Eq. (8), S-Eq. (44))	Real Wigner-D function (M-Eq. (13), S-Eq. (59))	$O(l^3)$	(M-Eq. (25), S-Def.4.2)	Real & complex Wigner-D function (M-Eq. (36), S-Prop.5.5)	$O(l^3)$
Light interaction	rendering equation	matrix-vector product (M-Eq. (5))	$O(l^4)$	polarized rendering equation	matrix-vector product (M-Eq. (41))	$O(4 \times 4l^4)$
+ Radial symmetry	convolution (M-Eq. (10), S-Def.2.11)	SH convolution (M-Eq. (11), S-Prop.2.13)	$O(l^2)$	polarized convolution (M-Eq. (63))	PSH convolution (M-Eq. (69), S-Eqs. (159,165,177))	$O(4 \times 4l^2)$
Visibility mask	Point-wise product	Triple product (M-Eq. (55), S-Eq. (145))	–	Point-wise product	Triple product (M-Eq. (56), S-Eq. (146))	–

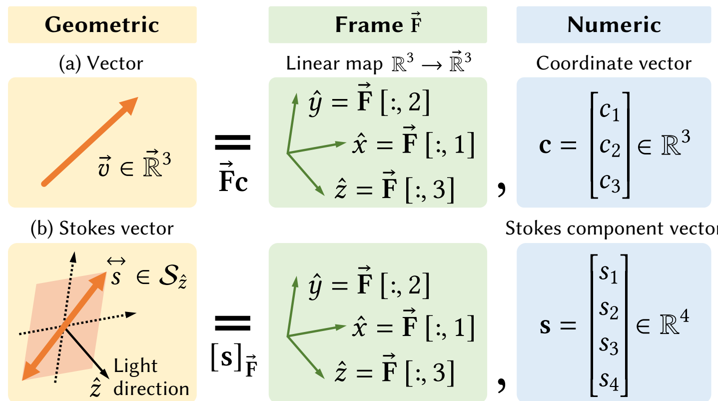

Fig. 1. We distinguish *geometric* and *numeric* quantities. (a) A (geometric) vector is equal to the product of an orthonormal *frame*, which is a linear map from numeric vectors to geometric vectors, and a coordinate vector, which is a kind of numeric vector. (b) Combining a frame $\vec{\mathbf{F}}$ and a numeric vector $\mathbf{s}$, named a *Stokes component vector*, we get a geometric quantity *Stokes vector*, which indicates a polarized intensity of a ray. Here, it is essentially different from the product of a frame and a numeric vector, we write the relationship of these quantities with our novel notation $\overleftrightarrow{s} = [\mathbf{s}]_{\vec{\mathbf{F}}}$.

Note that we also denote $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) =: \mathbb{R}^{m\times n}$. These multiplications are well defined in the sense of the action of linear maps on linear spaces and the composition of linear maps. Note that the multiplication of some pairs of quantities, which is not included above, is usually not allowed. For example to distinguish numeric matrices and geometric transforms, we can imagine a rotation. We denote $\mathbf{SO}(3) \subset \mathbb{R}^{3\times 3}$ and $\overrightarrow{SO}(3) \subset \mathcal{L}\left(\vec{\mathbb{R}}^3, \vec{\mathbb{R}}^3\right)$ as the sets of (numeric) rotation matrices and (geometric) rotation transforms, respectively. When a frame $\vec{\mathbf{F}} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \end{bmatrix}$ is given, the rotation transforms around the axis $\hat{x}$, $\hat{y}$, and $\hat{z}$ by angle θ can be written as follows:

$$\vec{R}_{\{\hat{x},\hat{y},\hat{z}\}} = \vec{\mathbf{F}}\mathbf{R}_{\{x,y,z\}}\vec{\mathbf{F}}^{-1}, \quad \text{where} \tag{2}$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathbf{R}_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

Note that while subscripts $\hat{x}$, $\hat{y}$, and $\hat{z}$ in the left-hand side of Equation 2 indicate the axis vectors of $\vec{\mathbf{F}}$ which has been defined in this context, subscripts x, y, and z in the right-hand side just symbols which means the first, second, and third of a frame which do not have to be given in advanced. Also note that conversion between a rotation transform $\vec{R} \in \overrightarrow{SO}(3)$ and $\mathbf{R} \in \mathbf{SO}(3)$ with respect to a frame $\vec{\mathbf{F}} \in \vec{\mathbb{F}}^3$ can be done by $\vec{R} = \vec{\mathbf{F}}\mathbf{R}\vec{\mathbf{F}}^{-1}$ and $\mathbf{R} = \vec{\mathbf{F}}^{-1}\vec{R}\vec{\mathbf{F}}$. For compactness, we often write consecutive rotation transforms about some axes $\hat{u}_1, \hat{u}_2, \cdots \in \hat{\mathbb{S}}^2$ and rotation matrices about $u_1, u_2, \cdots \in \{x, y, z\}$ as following ways, respectively:

$$\vec{R}_{\hat{u}_1}(\theta_1)\,\vec{R}_{\hat{u}_2}(\theta_2)\cdots =: \vec{R}_{\hat{u}_1\hat{u}_2\cdots}(\theta_1, \theta_2, \cdots), \quad \mathbf{R}_{u_1}(\theta_1)\,\mathbf{R}_{u_2}(\theta_2)\cdots =: \mathbf{R}_{u_1u_2\cdots}(\theta_1, \theta_2\cdots) \tag{4}$$

which also can represent Euler angles.

For numeric quantities, we will write NumPy style indexing notation such as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}[1] \\ \cdots \\ \mathbf{x}[n] \end{bmatrix}, \text{ and } \mathbf{A} = \begin{bmatrix} \mathbf{A}[1,1] & & \\ & \ddots & \\ & & \mathbf{A}[m,n] \end{bmatrix}. \tag{5}$$

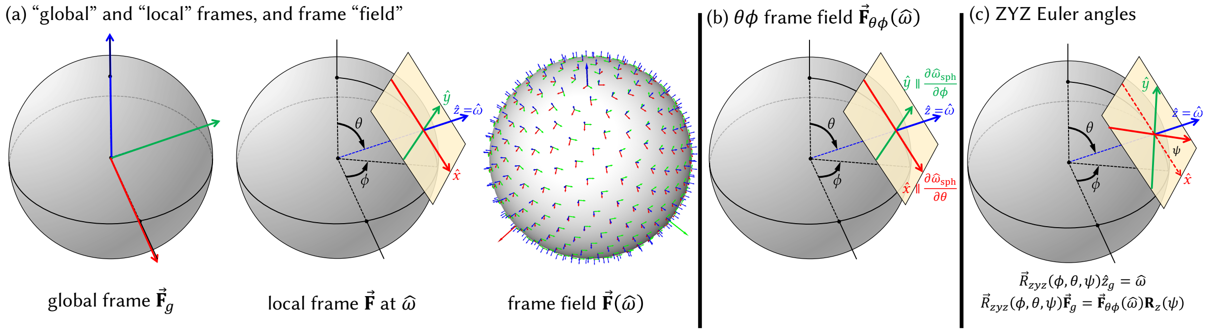

Fig. 2. Some definitions and useful identities about *frames*. (a) In this paper, we distinguish a *global frame*, a *local frame at* $\hat{\omega}$, and a *frame field*. (b) We usually use the $\theta\phi$ *frame field*, which is defined in Equation (9). (c) Spherical coordinates and a local frame can be evaluated with ZYZ Euler angles as Equation (12).

$\mathbf{A}[i,:]$ and $\mathbf{A}[:,i]$ denote i-th row and column vectors of a matrix $\mathbf{A}$. Referring to i-th (or i, j-th) entries of geometric quantities are illegal. Since a frame is both related to numeric and geometric vectors, referring its i-th row is illegal while its i-th column is well defined. For example, we have $\vec{\mathbf{F}}[:,3] = \hat{z}$ for a frame $\vec{\mathbf{F}} = \begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\end{bmatrix} \in \vec{\mathbb{F}}^3$.

Notations of sets of each type of quantity and notation convention for them are summarized in Table 1.

$$\mathbf{Mat}\left[A_{ijkl} \mid j = 1,\cdots,n, l = 1,\cdots,n\right] = \begin{bmatrix} A_{i1k1} & \cdots & A_{i1kn} \\ \vdots & \ddots & \vdots \\ A_{ink1} & \cdots & A_{inkn} \end{bmatrix}. \tag{6}$$

If the range of two indices is the same, then we sometimes write it as $j, l = 1, \cdots, n$ simply, and we sometimes even omit the range if it is clear in context. Moreover, we can also take the range of indices that is not an interval, such as:

$$\mathbf{Mat}\left[A_{ij} \mid i, j = +m, -m\right] = \begin{bmatrix} A_{+m,+m} & A_{+m,-m} \\ A_{-m,+m} & A_{-m,-m} \end{bmatrix}. \tag{7}$$

1.2 Unit Sphere, Frames, and Rotations

As a subset of the space of 3D geometric vectors $\vec{\mathbb{R}}^3$, the *unit sphere* (or just *sphere*) $\hat{\mathbb{S}}^2 = \left\{\hat{\omega} \in \vec{\mathbb{R}}^3 \mid \|\hat{\omega}\| = 1\right\}$[4] indicates the set of all vectors with unit norms. It also can be considered as the set of all directions in $\vec{\mathbb{R}}^3$ in the context of computer graphics. It usually parameterized by spherical coordinates of a zenith angle θ and a azimuth angle ϕ as follows:

$$\hat{\omega}_{\mathrm{sph}}(\theta,\phi) = \vec{\mathbf{F}}_g \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix}, \tag{8}$$

where a global frame $\vec{\mathbf{F}}_g = \begin{bmatrix}\hat{x}_g & \hat{y}_g & \hat{z}_g\end{bmatrix}$ is given.

In this paper, we will distinguish the terms global, local frames, and frame field as Figure 2(a). We call a *global frame* as a frame independent of a particular direction $\hat{\omega} \in \hat{\mathbb{S}}^2$, and the global frame, which is often used to assign spherical coordinates on $\hat{\mathbb{S}}^2$. A *local frame at* $\hat{\omega}$ indicates a frame with local $\hat{z}$ axis as $\hat{\omega}$, i.e., $\vec{\mathbf{F}}[:,3] = \hat{\omega}$, which is

[4]While denoting by $\mathbb{S}^2$ or S^2 is more common in other text, but we write with the ˆ symbol to clarify it is a set of geometric vectors, not numeric ones.

used to measure Stokes vectors along $\hat{\omega}$, and *frame field* (or moving frame) as a function from $\hat{\mathbb{S}}^2$ to $\vec{\mathbb{F}}^3$. We also define $\vec{\mathbb{F}}^3_{\hat{\omega}} := \left\{\vec{\mathbf{F}} \in \vec{\mathbb{F}}^3 \mid \vec{\mathbf{F}}[:,3] = \hat{\omega}\right\} \subset \vec{\mathbb{F}}^3$.

There are infinitely many choices to assign a frame field on the sphere $\hat{\mathbb{S}}^2$, we use a typical example which we call the $\theta\phi$ *frame field* and denote by $\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})$. Using spherical coordinates, it can be defined as follows:

$$\vec{\mathbf{F}}_{\theta\phi}(\theta,\phi) = \left[\frac{\partial\hat{\omega}_{\text{sph}}}{\partial\theta}(\theta,\phi) \quad \text{normalized}\frac{\partial\hat{\omega}_{\text{sph}}}{\partial\phi}(\theta,\phi) \quad \hat{\omega}_{\text{sph}}(\theta,\phi)\right], \tag{9}$$

which is visualized in Figure 2 (b). We observe that $\theta\phi$ frame field $\vec{\mathbf{F}}_{\theta\phi}$ has two singularities at $\hat{\omega}_{\text{sph}}(0,\phi) = \hat{z}_g$ and $\hat{\omega}_{\text{sph}}(\pi,\phi) = -\hat{z}_g$, where the function $\vec{\mathbf{F}}_{\theta\phi}$ cannot be continuously defined. Not only the $\theta\phi$ frame field, but any frame field on the sphere always has a singularity due to the hairy ball theorem, which is common in differential geometry.

1.3 Useful Identities

Identities using rotations. Note that inner products on $\mathbb{R}^3$ and $\vec{\mathbb{R}}^3$ are preserved under rotation. In other words,

$$\forall\mathbf{x},\mathbf{y}\in\mathbb{R}^3,\ \forall\mathbf{R}\in\mathbf{SO}(3),\quad \mathbf{x}^T\mathbf{y} = (\mathbf{R}\mathbf{x})^T(\mathbf{R}\mathbf{y}),\qquad \forall\vec{x},\vec{y}\in\vec{\mathbb{R}}^3,\ \forall\vec{R}\in\overrightarrow{SO}(3),\quad \vec{x}\cdot\vec{y} = \left(\vec{R}\vec{x}\right)\cdot\left(\vec{R}\vec{y}\right). \tag{10}$$

It can be directly proven by the fact that rotations are orthogonal matrix so that $\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I}$.

It is often useful that a global frame $\vec{\mathbf{F}}_g$ can also be considered as a local frame at the zenith ($\hat{z}_g$), and using ZYZ Euler angle rotation spherical coordinates and the $\theta\phi$ frame field can be rewritten as:

$$\hat{\omega}_{\text{sph}}(\theta,\phi) = \vec{R}_{\hat{z}_g\hat{y}_g\hat{z}_g}(\phi,\theta,\psi)\,\hat{z}_g, \tag{11}$$

$$\vec{\mathbf{F}}_{\theta\phi}(\theta,\phi) = \vec{R}_{\hat{z}_g\hat{y}_g\hat{z}_g}(\phi,\theta,0)\,\vec{\mathbf{F}}_g, \tag{12}$$

while $R_{\hat{z}_g\hat{y}_g\hat{z}_g}(\phi,\theta,\psi)\,\vec{\mathbf{F}}_g$ represents an arbitrary local frame at $\hat{\omega}_{\text{sph}}(\theta,\phi)$.

Another choice of a frame field is the perspective frame field $\vec{\mathbf{F}}_{\text{pers}}$ shown in Figure 5(c) in the main paper, characterized by the virtual perspective camera. Note that there are several choices of such camera-based frame field conventions. We follow the convention of Mitsuba 3 renderer [Jakob et al. 2022], which utilizes the up-axis of camera $\hat{u}$ to define $\vec{\mathbf{F}}_{\text{pers}}$ as

$$\vec{\mathbf{F}}_{\text{pers}}(\hat{\omega};\hat{u}) = \begin{bmatrix}\hat{x} & \hat{y} & \hat{\omega}\end{bmatrix} = \begin{bmatrix}\text{normalize}(\hat{u}\times\hat{\omega}) & \hat{\omega}\times\hat{x} & \hat{\omega}\end{bmatrix}. \tag{13}$$

While the $\theta\phi$ and the perspective frame fields are highly related, as $\vec{\mathbf{F}}_{\text{pers}}(\hat{\omega};\hat{z}_g) = \vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})\,\mathbf{R}_z\left(\frac{\pi}{2}\right)$, we use the both since they have their own convenience. Formulas of special functions, including SWSH and Wigner D-functions, are usually written related to $\theta\phi$ frame field, while it is natural to use perspective frame fields, whose local y axes are close to the camera up vector, for perspective view.

Integral formulae. To derive some identities for spherical harmonics and our polarized spherical harmonics, we sometimes need to integrate some functions over the space of rotation transforms $\overrightarrow{SO}(3)$. The differential measure $\mathrm{d}\vec{R}$ for $\vec{R}\in\overrightarrow{SO}(3)$ is evaluated as follows using ZYZ Euler angles with respect to a frame $\vec{\mathbf{F}} = \begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\end{bmatrix} \in \vec{\mathbb{F}}^3$ is given:

$$\int_{\overrightarrow{SO}(3)} f\left(\vec{R}\right)\mathrm{d}\vec{R} = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi} f\left(\vec{R}_{\hat{z}\hat{y}\hat{z}}(\alpha,\beta,\gamma)\right)\sin\beta\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma. \tag{14}$$

Note that this measure is equivalent to a constant multiple of the subspace measure by identifying $\overrightarrow{SO}(3)$ to a subset of $\mathbb{R}^9$, and it is invariant under choice of the frame $\vec{\mathbf{F}}$.

Several integration techniques for the sphere $\hat{\mathbb{S}}^2$ and rotation transforms $\overrightarrow{SO}(3)$ are used to prove the important properties of conventional and our polarized SH as:

$$\int_{\hat{\mathbb{S}}^2} f(\hat{\omega})\,\mathrm{d}\hat{\omega} = \frac{1}{2\pi}\int_{\overrightarrow{SO}(3)} f\left(\vec{R}\hat{z}_g\right)\mathrm{d}\vec{R}, \tag{15}$$

$$\int_{\hat{\mathbb{S}}^2} f(\hat{\omega})\,\mathrm{d}\hat{\omega} = \int_{\hat{\mathbb{S}}^2} f\left(\vec{R}\hat{\omega}\right)\mathrm{d}\hat{\omega} \text{ for any } \vec{R} \in \overrightarrow{SO}(3)\,. \tag{16}$$

Readers who are not about to verify the proof of this paper and just want to use the results can skip this part.

1.4 Linear Algebra on Function Spaces

We call an algebraic object equipped with addition and scalar multiplication as *linear space* while other literature more frequently calls it *vector space*. To avoid confusion, the term *vector* is usually used to discuss numeric vectors and geometric vectors in this paper.

This paper investigates several function spaces such as spherical harmonics, spin-weighted spherical harmonics, and naively applied spherical harmonics to Stokes vectors fields. To distinguish properties inherited from general properties of orthonormal basis and properties of a certain individual basis, we recall the general theory of linear algebra on function spaces in this section. Then, we will describe the properties of spherical harmonics as examples of general theory. Later, we introduce spin-weighted spherical harmonics in Section 6 in the main paper, also based on the language defined in this section.

First of all, We will discuss function spaces, including the set of spherical functions (or scalar fields on the sphere) $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ and the set of Stokes vector fields on the sphere in this paper. They are important in computer graphics since a spherical function can represent radiance as a function of directions, such as an environment map and a 2D slice of a BSDF with a fixed incoming or outgoing direction, and the set of Stokes vector fields can represent polarized versions of those quantities.

These function spaces are inner product spaces so they can be described by the general theory of linear algebra. Even though those function spaces have infinite dimensionality, fundamental properties of linear spaces are well extended to function spaces, as described in this section.

Spherical harmonics are known as *bases* of function spaces, so we first define bases and coefficient representation with respect to them.

Definition 1.1: Bases and coefficient vectors

For a countable index set I and an inner product space $\mathcal{H}$ (usually a function space) over scalar $\mathbb{K}$ (=$\mathbb{R}$ or $\mathbb{C}$), an indexed collection $\mathcal{B} := \{b_i \mid i \in I\} \subset \mathcal{H}$ is called a *basis* [5] of $\mathcal{H}$ if and only if for any $f \in \mathcal{H}$ there uniquely exists an indexed collection of scalars $\{a_i \mid i \in I\}$ such that:

$$f = \sum_{i \in I} \mathrm{f}_i b_i. \tag{17}$$

Here, a_i is called the *coefficient of f with respect to b_i* or the *i-th coefficient of f with respect to $\mathcal{B}$*. When order on I is given in context, $\mathbf{f} := \mathbf{Mat}\left[\mathrm{f}_i \mid i \in I\right]$ is called the *coefficient vector* of f with respect to the basis $\mathcal{B}$.

While bases are usually defined without admitting such index sets as above, having them in the definition of bases makes writing statements about spherical harmonics and further bases, including our polarized spherical

[5] Rigorously, it should be called a Hilbert basis since Equation (17) includes not only finite summations but also countably infinite ones, but we simply call basis for simplicity.

harmonics, much more convenient. Note that converting a vector f into its coefficient vector is linear, so the coefficient vector can be considered to be equivalent to the original vector f. For the sake of simplicity, we consistently denote I, $\mathcal{H}$, $\mathbb{K}$, and $\mathcal{B}$ with the conditions stated in Definition 1.1.

In Section 1.4 italic characters such as f and b_i usually denote elements of a vector space, which also can be functions, Roman characters such as f_i denote coefficients with respect to some basis, and bold roman characters such as $\mathbf{f}$ do coefficient vectors. While f may be a geometric or numeric quantity depending on its space $\mathcal{H}$, f_i, and $\mathbf{f}$ can be considered as numeric quantities since they are indexed collections of scalars.

Proposition 1.2: Coefficient for a basis

In Definition 1.1, suppose that $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ denotes the inner product on $\mathcal{H}$ and the basis $\{b_i \mid i \in I\}$ is orthonormal, i.e., $\left\langle b_i, b_j\right\rangle_{\mathcal{H}} = \delta_{ij}$. Then the coefficient in Equation (17) is evaluated as $\mathrm{f}_i = \langle b_i, f\rangle_{\mathcal{H}}$, i.e.,

$$\forall f \in \mathcal{H}, \quad f = \sum_{i\in I} \langle b_i, f\rangle_{\mathcal{H}}\, b_i. \tag{18}$$

Definition 1.3: projection on subsets of bases

Suppose that $\mathcal{B}' = \{b_i \mid i \in J\}$ be a subset of a basis $\mathcal{B}$ of a linear space $\mathcal{H}$, which is characterized by $J \subset I$. A *projection of* $f \in \mathcal{H}$ *on* $\mathcal{B}'$ is defined as:

$$\tilde{f} = \sum_{i\in J} \mathrm{f}_i b_i,$$

where f_i is the coefficient of f with respect to b_i. This is also called the projection of f on the basis of $\mathcal{B}$ *up to* J.

Note that the projection of f up to J sometimes indicates the coefficients $\{\mathrm{f}_i \mid i \in J\}$ rather than $\sum_{i\in J} \mathrm{f}_i b_i$, and it will be distinguished according to the context.

Note that the space of linear map $\mathcal{L}(X, Y)$ is still well defined even if X is an infinite-dimensional function space. However, in this case, we usually call such linear maps as *linear operators* conventionally to emphasize that X may be a function space.

Proposition 1.4: Coefficient matrices of linear operators

Suppose that $\{b_i \mid i \in I\} \subset \mathcal{H}$ is an orthonormal basis, there is a linear operator $T : \mathcal{H} \to \mathcal{H}$. When denoting the coefficients of $f \in \mathcal{H}$ and $T[f]$ by $\{\mathrm{f}_i \mid i \in I\}$ and $\left\{\mathrm{f}'_i \mid i \in I\right\}$, respectively, f'_i is evaluated as:

$$\mathrm{f}'_i = \sum_{j\in I} \left\langle b_i, T\left[b_j\right]\right\rangle_{\mathcal{H}} \mathrm{f}_j. \tag{19}$$

Here, $\left\langle b_i, T\left[B_j\right]\right\rangle$ is called as the *coefficient of* T *with respect to* $\left(b_i, b_j\right)$ or *(i,j)-th coefficient of the linear operator* T with respect to the basis $\mathcal{B}$. When order on I is given in context, $\mathbf{T} \coloneqq \mathbf{Mat}\left[\left\langle b_i, T\left[B_j\right]\right\rangle_{\mathcal{H}} \mid i, j \in I\right]$ is called the *coefficient matrix* of T *with respect to* $\mathcal{B}$.

Proof: By Proposition 1.2,

$$\mathrm{f}_i = \langle b_i, f \rangle_{\mathcal{H}}, \text{ and } \mathrm{f}'_i = \langle b_i, T[f] \rangle_{\mathcal{H}}.$$

From the later equation, substituting the formal equation and the definition of basis (Equation (1.1)) yields:

$$\mathrm{f}'_i = \langle b_i, T[f] \rangle_{\mathcal{H}} = \left\langle b_i, T\left[\sum_{j \in I} \mathrm{f}_j b_j\right]\right\rangle_{\mathcal{H}} = \sum_{j \in I} \left\langle b_i, T\left[b_j\right]\right\rangle_{\mathcal{H}} \mathrm{f}_j.$$

Here, the rightmost implication comes from the linearity of T and the inner product. □

Note that Equation (19) can be rewritten as the matrix-vector product of the coefficient matrix of T and the coefficient vector of f. For coefficients of linear operators, the following properties are useful.

Proposition 1.5: Identities for linear operator coefficients

Suppose that $\mathcal{B} = \{b_i \mid i \in I\} \subset \mathcal{H}$ is an orthonormal basis on $\mathcal{H}$ and there are linear operators $T, T_1, T_2 : \mathcal{H} \to \mathcal{H}$. Denote their coefficient matrices with respect to $\mathcal{B}$ by $\mathbf{T}$, $\mathbf{T}_1$, and $\mathbf{T}_2$. The following properties hold.

(1) For the identity operator $I : \mathcal{H} \to \mathcal{H}$ with $I[f] = f$, the coefficient matrix w.r.t. $\mathcal{B}$ is the identity matrix, i.e., $\left\langle b_i, I\left[b_j\right]\right\rangle_{\mathcal{H}} = \delta_{ij}$.

(2) The coefficient matrix of $T_1 \circ T_2$ w.r.t. $\mathcal{B}$ is the matrix product of $\mathbf{T}_1$ and $\mathbf{T}_2$, i.e.:

$$\sum_{k \in I} \langle b_i, T_1[b_k] \rangle_{\mathcal{H}} \left\langle b_k, T_2\left[b_j\right]\right\rangle_{\mathcal{H}} = \left\langle b_i, T_1 \circ T_2\left[b_j\right]\right\rangle_{\mathcal{H}}.$$

(3) If T^{-1} exists, then the coefficient matrix of T^{-1} w.r.t. $\mathcal{B}$ is the inverse matrix of $\mathbf{T}$, i.e.:

$$\sum_{k \in I} \langle b_i, T[b_k] \rangle_{\mathcal{H}} \left\langle b_k, T^{-1}\left[b_j\right]\right\rangle_{\mathcal{H}} = \sum_{k \in I} \left\langle b_i, T^{-1}[b_k]\right\rangle_{\mathcal{H}} \left\langle b_k, T\left[b_j\right]\right\rangle_{\mathcal{H}} = \delta_{ij}$$

(4) If T is a symmetric operator, i.e., $\langle f, T[g] \rangle_{\mathcal{H}} = \langle T[f], g \rangle_{\mathcal{H}}$ for any $f, g \in \mathcal{H}$, then its coefficient matrix $\mathbf{T}$ is a Hermitian matrix ($\mathbf{T}^T = \mathbf{T}^*$), i.e., $\left\langle b_i, T\left[b_j\right]\right\rangle_{\mathcal{H}} = \left\langle b_j, T[b_i]\right\rangle^*_{\mathcal{H}}$.

(5) If T is a unitary operator, i.e., $\langle f, T[g] \rangle_{\mathcal{H}} = \left\langle T^{-1}[f], g\right\rangle_{\mathcal{H}}$ for any $f, g \in \mathcal{H}$, then its coefficient matrix $\mathbf{T}$ is a unitary matrix ($\mathbf{T}^{-1} = \left(\mathbf{T}^T\right)^*$), i.e., $\left\langle b_i, T^{-1}\left[b_j\right]\right\rangle_{\mathcal{H}} = \left\langle b_j, T[b_i]\right\rangle^*_{\mathcal{H}}$.

What we deal with as an important desirable property of spherical harmonics is rotation invariance. For a generalized description, we first formulate *transform invariance* for given transforms and discuss the rotation invariance of spherical harmonics in the later section. First, the invariance of a subset of a space is naturally defined.

Definition 1.6: Transform invariance of a subset

A set $A \subset \mathcal{H}$ is called to be invariant under a linear operator $T : \mathcal{H} \to \mathcal{H}$ if $T(A) = A$.

Here, we also call the linear operator T as a *transform* conventionally when we are interested in invariance.

Now, a basis can be called to be invariant if it can be separated into a partition of finite sets so that these finite subsets of the basis span invariant subspaces.

Definition 1.7: Transform invariance

A basis $\{b_i \mid i \in I\} \subset \mathcal{H}$ is called to be invariant under a linear operator (transform) $T : \mathcal{H} \to \mathcal{H}$ if there exists a partition of the index set I into finite subsets, i.e., $I = \bigcup_{k=0}^{\infty} J_k$ with $J_i \cap J_j = \emptyset$, such that span $\{b_i \mid i \in J_k\}$ is invariant under T for any k.

Proposition 1.8: Equivalent conditions for transform invariance

Suppose that there is an orthonormal basis $\mathcal{B} = \{b_i \mid i \in I\}$ of an inner product space $\mathcal{H}$ and a linear operator (transform) $T : \mathcal{H} \to \mathcal{H}$. Then, the following statements are equivalent to each other.

(i) The basis is invariant under T, by Definition 1.7.

(ii) Let $\mathcal{B}_k := \{b_i \mid i \in J_k\}$. For any $f \in \mathcal{H}$ and $k \geq 0$ the projection of $T[f]$ on $\mathcal{B}_k$ is equal to $T[f']$ where f' is the projection of f on $\mathcal{B}_k$.

(iii) Let $\mathcal{B}_{\leq k} := \left\{b_i \mid i \in J_j \text{ for some } j \leq k\right\}$. For any $f \in \mathcal{H}$ and $k \leq 0$ the projection of $T[f]$ on $\mathcal{B}_{\leq k}$ is equal to $T[f']$ where f' is the projection of f on $\mathcal{B}_{\leq k}$.

(iv)
$$\forall k \neq k' \geq 0,\ \forall (i,j) \in J_k \times J_{k'},\ \left\langle b_i, T\left[B_j\right]\right\rangle_{\mathcal{H}} = 0. \tag{20}$$

Proof: For simplicity, we will briefly show a few implications among (i) and (iv) rather than full proof.
(i) $\Longrightarrow$ (iv): For $i \in J_k$, there exist some a_{ij} for $j \in J_k$ such that $T[b_i] = \sum_{j \in J_k} a_{ij} b_j$ since $T[b_i] \in$ span $\{b_i \mid i \in J_k\}$ by invariance in Definition 1.7. Since $\mathcal{B}$ is an orthonormal basis of $\mathcal{H}$, we can rewrite: $T[b_i] = \sum_{i \in I} \left\langle b_j, T[b_i]\right\rangle_{\mathcal{H}} b_j$. Note that basis yields the unique linear coefficients so that we finally get $a_{ij} = \left\langle b_j, T[b_i]\right\rangle_{\mathcal{H}}$ for $j \in J_k$ and $\left\langle b_j, T[b_i]\right\rangle_{\mathcal{H}} = 0$ for $j \notin J_k$. The latter one implies P_4.
(iv) $\Longrightarrow$ (iii): Note that $f = \sum_{i \in I} \langle b_i, f\rangle_{\mathcal{H}} b_i$ by Equation (18). By linearity of T, we get $T[f] = \sum_{i \in I} \langle b_i, f\rangle_{\mathcal{H}} T[b_i]$. Expanding $T[b_i]$ using Equation (18) yields:

$$T[f] = \sum_{i \in I} \sum_{j \in I} \langle b_i, f\rangle_{\mathcal{H}} \left\langle b_j, T[b_i]\right\rangle_{\mathcal{H}} b_j. \tag{21}$$

Since it is a linear combination of basis b_j, $\sum_{i \in I} \langle b_i, f\rangle_{\mathcal{H}} \left\langle b_j, T[b_i]\right\rangle_{\mathcal{H}}$ is the coefficient of $T[f]$ w.r.t. b_j. Changing letters for summation indices and using (iv), we finally get the projection of $T[f]$ on $\mathcal{B}_{\leq k}$ is:

$$\sum_{j \leq k, i' \in J_j} \sum_{i \in I} \langle b_i, f\rangle_{\mathcal{H}} \langle b_{i'}, T[b_i]\rangle_{\mathcal{H}} b_{i'} = \sum_{j \leq k, i' \in J_j} \sum_{i \in J_j} \langle b_i, f\rangle_{\mathcal{H}} \langle b_{i'}, T[b_i]\rangle_{\mathcal{H}} b_{i'}. \tag{22}$$

On the other hand:

$$f' = \sum_{j \leq k, i \in J_j} \langle b_i, f\rangle_{\mathcal{H}} b_i, \tag{23}$$

$$\begin{aligned} T[f'] &= \sum_{j \leq k, i \in J_j} \langle b_i, f\rangle_{\mathcal{H}} T[b_i] = \sum_{j \leq k, i \in J_j} \sum_{i' \in I} \langle b_i, f\rangle_{\mathcal{H}} \langle b_{i'}, T[b_i]\rangle_{\mathcal{H}} b_{i'} \\ &= \sum_{j \leq k, i \in J_j} \sum_{i' \in J_j} \langle b_i, f\rangle_{\mathcal{H}} \langle b_{i'}, T[b_i]\rangle_{\mathcal{H}} b_{i'}. \end{aligned} \tag{24}$$

Here, the last implication comes from (iv). Now we observe that Equations (22) and (24) are equal.

(iii) $\Longrightarrow$ (ii): It is straightforward since a projection is a linear operation and a projection on $\mathcal{B}_k$ is identical to the subtraction of the projection on $\mathcal{B}_{\leq k-1}$ from that on $\mathcal{B}_{\leq k}$ □

We observe here the matrix representation of $\langle b_i, T[b_j] \rangle$ satisfying Equation (20) becomes a block diagonal matrix. Conditions (i) and (iv) can be determined only with the basis and the transform themselves, while (ii) and (iii) show why this invariance is important when one applies the transform for a given vector. In other words, for an invariant basis and a transform, projecting on a subset of the basis and applying the transform commute without any loss of information. Moreover, this commutativity allows us to reduce the transform applied on a projected vector to a finite computation even if the vector space $\mathcal{H}$ has an infinite dimensionality.

Proposition 1.9: Finite matrix for invariant transform

There is an orthonormal basis $\mathcal{B} = \{b_i \mid i \in I\}$ of an inner product space $\mathcal{H}$ and a linear operator $T : \mathcal{H} \to \mathcal{H}$. Suppose that $\mathcal{B}$ is invariant under T with a partition of finite indices $I = \bigcup_{k=0}^{\infty} J_k$. For any $f \in \mathcal{H}$, let $\mathbf{f}_{\leq k} := \{\langle b_i, f \rangle \mid b_i \in \mathcal{B}_{\leq k}\}$ denote the (finite) coefficient vector of f projected onto $\mathcal{B}_{\leq k} := \{b_i \mid j \leq k,\ i \in J_j\}$. Then the $T[f]$ projection on $\mathcal{B}_{\leq k}$ is evaluated as the following finite matrix-vector product.

$$\mathbf{T}_{\leq k}\mathbf{f}_{\leq k}, \text{ where } \mathbf{T}_{\leq k} := \mathbf{Mat}\left[\langle b_i, T[b_j] \rangle \mid b_i, b_j \in \mathcal{B}_{\leq k}\right]. \tag{25}$$

Proposition 1.9 is a necessary but not sufficient condition of invariance described in Definition 1.7 and Proposition 1.8, but it is related to what actually a rendering pipeline computes. Thus, Figures 10 and 11 in our main paper shows experimental validation of Proposition 1.9.

1.4.1 Linear spaces over $\mathbb{R}$ vs. $\mathbb{C}$. In this paper, we sometimes consider a linear space with the scalar as $\mathbb{R}$ and sometimes do so with the scalar as $\mathbb{C}$. Then some relations discussed here will be useful.

Proposition 1.10: Linear spaces over $\mathbb{R}$ vs. $\mathbb{C}$

If $\mathcal{B}$ is a basis for a linear space V over $\mathbb{C}$ then the set $\mathcal{B}' := \{e, ie \mid e \in \mathcal{B}\}$ is a basis for V as a linear space over $\mathbb{R}$. Concretely, if an arbitrary vector $v \in V$ is represented as a linear combination over complex coefficients by Equation (17) as:

$$v = \sum_i c_i e_i, \tag{26}$$

then it can be rewritten using the new basis $\mathcal{B}'$ and real coefficients as follows:

$$v = \sum_i a_i e_i + b_i (ie_i), \quad \text{where} \quad a_i := \Re c_i \quad \text{and} \quad b_i := \Im c_i. \tag{27}$$

Moreover, V over $\mathbb{C}$ is equipped with an inner product $\langle \cdot, \cdot \rangle_{V|\mathbb{C}}$, an inner product on V over $\mathbb{R}$ is canonically induced as $\langle \cdot, \cdot \rangle_{V|\mathbb{R}} := \Re \langle \cdot, \cdot \rangle_{V|\mathbb{C}}$. $\mathcal{B}$ is orthonormal (w.r.t. $\langle \cdot, \cdot \rangle_{V|\mathbb{C}}$) then the new basis $\mathcal{B}'$ is orthonormal with respect to $\langle \cdot, \cdot \rangle_{V|\mathbb{R}}$.

Here, $\Re$ and $\Im$ denote taking real and imaginary parts of given complex numbers, respectively. We often write each inner product as $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}}$, respectively, for simplicity when it is clear in context. The following relations between coefficients and bases are useful.

$$c_i = \langle B_i, v\rangle_{\mathbb{C}}, \tag{28}$$

$$a_i = \Re \langle B_i, v\rangle_{\mathbb{C}} = \langle B_i, v\rangle_{\mathbb{R}}, \tag{29}$$

$$b_i = \Re \langle iB_i, v\rangle_{\mathbb{C}} = \langle iB_i, v\rangle_{\mathbb{R}} = \Im \langle B_i, v\rangle_{\mathbb{C}}. \tag{30}$$

Please be careful that while B_i and iB_i are not orthogonal in V over $\mathbb{C}$ (i.e., $\langle B_i, iB_i\rangle_{\mathbb{C}} = i \neq 0$), these are orthogonal in V over $\mathbb{R}$ (i.e., $\langle B_i, iB_i\rangle_{\mathbb{R}} = 0$).

2 BACKGROUND: SPHERICAL HARMONICS

2.1 Spherical Harmonics

Spherical harmonics is described as a special case of Definition 1.1, as:

Proposition 2.1: Spherical harmonics

Spherical harmonics are spherical functions $Y_{lm} \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ which can be evaluated in spherical coordinates (θ, ϕ) as follows:

$$Y_{lm}(\theta,\phi) = A_{lm} P_l^m(\cos\theta)\, e^{im\phi}, \tag{31a}$$

$$A_{lm} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}, \qquad P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}\left(x^2-1\right)^l, \tag{31b}$$

$$P_l^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x), \qquad P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x), \text{ for } m \geq 0.$$

With an index set $I_{\mathrm{SH}} = \left\{(l,m) \in \mathbb{Z}^2 \mid |m| \leq l\right\}$, $\{Y_{lm} \mid (l,m) \in I_{\mathrm{SH}}\}$ is an orthonormal basis of $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$.

Here, P_l is called the *Legendre function of order l* and P_l^m is called the *associated Legendre function of order l and degree m.* [6] The first few spherical harmonics functions can easily be evaluated using the recurrence relations above as follows.

$$\begin{aligned}
&Y_{00}(\theta,\phi) = \sqrt{\frac{1}{4\pi}}, && Y_{1,-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}, && Y_{10}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta,\\
&Y_{11}(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}, && Y_{2,-2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-2i\phi}, && Y_{2,-1}(\theta,\phi) = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\phi},\\
&Y_{20}(\theta,\phi) = \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta - 1\right), && Y_{21}(\theta,\phi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}, && Y_{22}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}
\end{aligned} \tag{32}$$

Be careful that other literature and programming libraries sometimes use different conventions in Equation (31), so that they might have slightly different formulae such as multiplying $(-1)^m$ or $\sqrt{4\pi}$.

Orthonormality defined in Proposition 1.2 assumes the set of spherical functions $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ as an inner product space. An inner product of two spherical functions f and $g \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ is an integral of the product of the values of the given two functions in each direction:

$$\langle Y_{lm}, f\rangle_{\mathcal{F}(\hat{\mathbb{S}}^2,\mathbb{C})} = \int_{\hat{\mathbb{S}}^2} f^*(\hat{\omega})\, g(\hat{\omega})\, \mathrm{d}\omega, \tag{33}$$

[6] Unfortunately, there is a difference in terminologies *order* and *degree* depending on each research field. Mathematics and physics such as [Canzani 2013; Hall 2013] usually call l and m by *degree* and *order* respectively. We follow computer graphics convention as [Ramamoorthi and Hanrahan 2001a; Sloan et al. 2002; Xin et al. 2021].

where $\mathrm{d}\hat{\omega} = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the solid angle measure on the sphere $\hat{\mathbb{S}}^2$. Note the presence of the complex conjugation, whereas it can be ignored for real-valued functions. Note that inner products on other function spaces can be defined in a similar way in Section 5.

Applying Equation (17) in Definition 1.1 and Proposition 1.2 implies that any spherical function $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ is equal to an infinite number of linear combination of spherical harmonics as:

$$f = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathrm{f}_{lm} Y_{lm}, \tag{34}$$

and the *coefficient* f_{lm} is computed as

$$\mathrm{f}_{lm} = \langle Y_{lm}, f \rangle_{\mathcal{F}(\hat{\mathbb{S}}^2, \mathbb{C})} . \tag{35}$$

An infinite dimensional numeric vector $\left[\mathrm{f}_{00}, \mathrm{f}_{1,-1}, \mathrm{f}_{10}, \mathrm{f}_{11}, \cdots\right]^T$, which is called the *coefficient vector* of f, represents continuously defined f without loss of information. However, we can take the *projection of f on spherical harmonics up to $l = l_{\max}$* by Definition 1.3 so that store it into a finite numeric vector $\left[\mathrm{f}_{00} \quad \cdots \quad \mathrm{f}_{l_{\max}, l_{\max}}\right]^T$ of $(l_{\max}+1)^2 = O\left(l_{\max}^2\right)$ entries. It can also be understood as a *smoothed* data of f up to the $l_{\max}$-th frequency band.

We observe that spherical harmonics satisfy the following identities, which will be used later.

Proposition 2.2: Spherical harmonics identities

$$Y_{lm}^* = (-1)^m \, Y_{l,-m} \tag{36}$$

$$Y_{lm}(\hat{\omega}) = (-1)^{l+m} \, Y_{lm}(-\hat{\omega}) \tag{37}$$

2.1.1 Zonal harmonics. There is an important subset of spherical harmonics, which is useful for spherical functions with some symmetry. When a global frame $\vec{\mathbf{F}}_g$ is fixed, a spherical function $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is called to be *azimuthally (axially) symmetric* if $f\left(\vec{R}_{\hat{z}_g}(\alpha)\,\hat{\omega}\right) = f(\hat{\omega})$ for any $\alpha \in \mathbb{R}$ and $\hat{\omega} \in \hat{\mathbb{S}}^2$. Note that such a function can be simply written as $f(\theta)$, a function of the single zenith angle θ. Note that the two formulations of an azimuthally symmetric function about $\theta \in [0, \pi]$ and $\hat{\omega} \in \hat{\mathbb{S}}^2$, respectively, are interchangeable using the following relation.

$$\underbrace{f(\theta)}_{\text{domain } [0,\pi]} = f\left(\cos^{-1} \hat{z}_g \cdot \hat{\omega}\right), \tag{38a}$$

$$\underbrace{f(\hat{\omega})}_{\text{domain } \hat{\mathbb{S}}^2} = f\left(\hat{\omega}_{\mathrm{sph}}(\theta, \phi)\right), \text{ with any choice of } \phi \in \mathbb{R}. \tag{38b}$$

Spherical harmonics Y_{l0} with zero degrees ($m = 0$) is called *Zonal harmonics*, and the set of Zonal harmonics is a basis of the space of azimuthally symmetric spherical functions. In other words, Y_{l0} has azimuthal symmetry, and conversely any function $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ can be represented as $f = \sum_{l=0}^{\infty} \mathrm{f}_{l0} Y_{l0}$. Note that in contrast to SH Y_{lm} for $m \neq 0$, Zonal harmonics basis Y_{l0} always has real values so that f_{l0} is also real for any real-valued function $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{R}\right)$.

2.2 Linear Operators in Spherical Harmonics

First, let's investigate the desirable properties of linear operators on spherical functions.

Definition 2.3: Linear operators and kernels

Suppose there is a function $K \in \mathcal{F}\left(\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2, \mathbb{K}\right)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The *linear operator with the kernel* K, denoted by $K_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)\right)$, is defined as follows:

$$\forall f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right), \quad K_{\mathcal{F}}\left[f\right](\hat{\omega}_i) = \int_{\hat{\mathbb{S}}^2} K(\hat{\omega}_i, \hat{\omega}_o) f(\hat{\omega}_i)\, \mathrm{d}\hat{\omega}_i. \tag{39}$$

If a linear operator $K_{\mathcal{F}}$ was given first, a function K satisfying the above equation is called the *operator kernel* (or simply *kernel*) of the operator $K_{\mathcal{F}}$.

Here, we slightly abuse the notation of the symbol $\mathcal{F}$. While on the first page $\mathcal{F}(X, Y)$ is defined as the set of functions from X to Y for given sets X and Y, in Definition 2.3 $K_{\mathcal{F}}$ denotes a *functional version* of the given K. Note that we will define such functional versions of a given quantity in different ways depending on the type of the given quantity. While such different ways will share the notation of the subscript $\mathcal{F}$ in this paper, they will be clearly distinguished in context.

In Section 2.2, we usually call the operator kernels simply *kernels*, but in later sections, we often refer to them as *operator kernel* to distinguish them from convolution kernels which will be introduced in Section 2.6.

As a special case of Proposition 19, spherical harmonics provide frequency-domain formulations of spherical functions and linear operators on these spherical functions.

In the context of computer graphics, while a spherical function can be radiance as a function of directions, including an environment map, a linear operator on spherical functions can be a light interaction effect.

One of the simplest cases of it is surface reflection determined by a bidirectional reflectance distribution function (BRDF). Assuming we have a BRDF $\rho\colon \hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2 \to \mathbb{R}$, its surface reflection can be considered as a linear operator $\rho^{\perp}_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)\right)$ which maps incident radiance to outgoing radiance through the rendering equation as follows:

$$\rho^{\perp}_{\mathcal{F}}\left[L^{\text{in}}\right](\hat{\omega}_o) = L^{\text{out}} = \int_{\hat{\mathbb{S}}^2} \rho(\hat{\omega}_i, \hat{\omega}_o) \left|\hat{n} \cdot \hat{\omega}_i\right| L^{\text{in}}(\hat{\omega}_i)\, \mathrm{d}\hat{\omega}_i, \tag{40}$$

where the superscript $\perp$ denotes cosine-weighted.[7] Not only reflection due to a BRDF, other light interaction effects, including self-shadowing and self-transfer, can also be described as linear operators in similar ways by replacing $\rho(\hat{\omega}_i, \hat{\omega}_o) \left|\hat{n} \cdot \hat{\omega}_i\right|$ to other functions.

Once we have a linear operator $\rho^{\perp}_{\mathcal{F}}$, we can convert both the operator itself and the evaluation of the operator on a spherical function into frequency domain formulation using spherical harmonics. First, the *coefficient of* $\rho^{\perp}_{\mathcal{F}}$ *with respect to* $\left(Y_{l_o m_o}, Y_{l_i m_i}\right)$ or the $(l_o, m_o) - (l_i, m_i)$*-th coefficient of* $\rho^{\perp}_{\mathcal{F}}$ with respect to SH is defined as:

$$\rho_{l_o m_o, l_i m_i} \coloneqq \left\langle Y_{l_o m_o}, \rho^{\perp}_{\mathcal{F}}\left[Y_{l_i m_i}\right]\right\rangle_{\mathcal{F}}. \tag{41}$$

Considering each pair of indices $(l, m) \in I_{\text{SH}}$ to be linearly enumerated, Equation (41) converts the linear operator $\rho^{\perp}_{\mathcal{F}}$ into a (either finite or infinite) numeric matrix with the elements $\rho_{l_o m_o l_i m_i}$ in the (l_o, m_o)-th row and the (l_i, m_i)-th column, called the *coefficient matrix* of $\rho^{\perp}_{\mathcal{F}}$.

In the case of the operator $\rho^{\perp}_{\mathcal{F}}$, it has a kernel. Then, the coefficient can also evaluated from the kernel as follows.

$$\rho_{l_o m_o, l_i m_i} = \int_{\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2} Y^*_{l_o m_o}(\hat{\omega}_o)\, \rho^{\perp}(\hat{\omega}_i, \hat{\omega}_o)\, Y_{l_i m_i}(\hat{\omega}_i)\, \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o. \tag{42}$$

[7]Note that in our main paper, we assume that the notation ρ denotes a cosine-weighted BRDF for the sake of simplicity

Then the rendering equation in Equation (40) is reformulated as the following by Equation (19):

$$\mathrm{L}^{\mathrm{out}}_{l_o m_o} = \sum_{(l_i,m_i)\in I_{\mathrm{SH}}} \rho_{l_o m_o, l_i m_i} \mathrm{L}^{\mathrm{in}}_{l_i m_i}, \tag{43}$$

where $\mathrm{L}^{\{\mathrm{in,out}\}}_{lm} := \left\langle Y_{lm}, L^{\{\mathrm{in,out}\}} \right\rangle$ denotes the (l, m)-th SH coefficient of incident and outgoing radiance, respectively.

Note that the above equation can be considered as a matrix multiplication with the integer pairs (l_o, m_o) as rows and the integer pairs (l_i, m_i) as columns. Figure 4 illustrates a coefficient matrix of a linear operator (Equation (41)) and how its action on a spherical function (Equation (43)) can be converted into a matrix-vector product in the SH coefficient domain. Note that the special case of the given linear operator in Figure 4, including its sparsity, will be explained in the next subsection.

Taking finite coefficients up to orders $l \le l_{\max}$, the SH coefficient matrix of a linear operator consists of $(l_{\max} + 1)^4 = O\,(l_{\max})$ in general, since it consists of $(l_{\max} + 1)^2$ rows and columns.

Encoding linear operators into coefficient matrices as described in this subsection follows directly from the general theory described in Section 1.4, so it can be applied in a similar way to other types of basis in a similar way. However, the strengths of SH appear when investigating sparsity and analytic formulations for coefficient matrices of special kinds of linear operators. In the next subsections, except for Section 2.4, we will investigate coefficient matrices of the functional version of rotation transforms (Section 2.3), operators with azimuthal symmetry (isotropic BRDF, Section 2.5) and rotation equivariance (Section 2.6), and the functional version of the reflection operation which flips a direction vector to its antipodal direction (Section 2.7). Note that the main theoretical purpose of this paper is to extend the desirable properties found in these subsections to the domain of a novel basis introduced in Section 5 taking Mueller calculus (Section 3) into account.

Application in precomputation-based rendering. When the SH coefficient vector of L^{in} and the SH coefficient matrix of $\rho^{\perp}_{\mathcal{F}}$ have been precomputed, environment map lighting can be computed efficiently as a matrix-vector product in runtime [Ramamoorthi and Hanrahan 2001a]. In the precomputed radiance transfer (PRT) methods, the coefficient matrix, which is also called the *radiance transfer matrix*, can contain further light transport effects, such as self-shadowing and inter-reflection, by replacing $\rho\,(\hat{\omega}_i, \hat{\omega}_o)$ in precomputation time [Sloan et al. 2002]. In particular, self-shadowing can be achieved by replacing $\rho\,(\hat{\omega}_i, \hat{\omega}_o)$ with $\rho\,(\hat{\omega}_i, \hat{\omega}_o)\, V\,(\hat{\omega}_i, \hat{\omega}_o)$, where V is the binary visibility function.

2.3 Rotation of Spherical Harmonics

One of the most important properties of spherical harmonics, which is not satisfied by another basis, such as spherical wavelets and spherical Gaussian, is rotation invariance. We first formulate how a rotation transform can act on *functions*, not only individual vectors, and then investigate the rotation of spherical harmonics.

First of all, given a rotation transform $\vec{R} \in \overrightarrow{SO}\,(3)$, which is a function from $\vec{\mathbb{R}}^3$ onto $\vec{\mathbb{R}}^3$ (restricted to a function from $\hat{\mathbb{S}}^2$ to $\hat{\mathbb{S}}^2$), we naturally define a rotation of *functions*, denoted by $\vec{R}_{\mathcal{F}}$, as follows:

$$\vec{R}_{\mathcal{F}} : \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right) \to \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right), \quad \vec{R}_{\mathcal{F}}\,[f]\,(\hat{\omega}) = f\left(\vec{R}^{-1}\hat{\omega}\right), \tag{44}$$

where this rotation on functions is also described in Figure 3 (a), and $\vec{R}_{\mathcal{F}}$ can also be considered as functions on real-valued functions, i.e., $\vec{R}_{\mathcal{F}} : \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{R}\right) \to \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{R}\right)$.

We observe that Equation (44) is linear about f, so $\vec{R}_{\mathcal{F}}$ is a linear operator on the space of spherical functions $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$. Then, we can formulate the rotation invariance of spherical harmonics in the same manner as Definition 1.7. Using (iv) in Proposition 1.8, we can formulate the invariance property as follows:

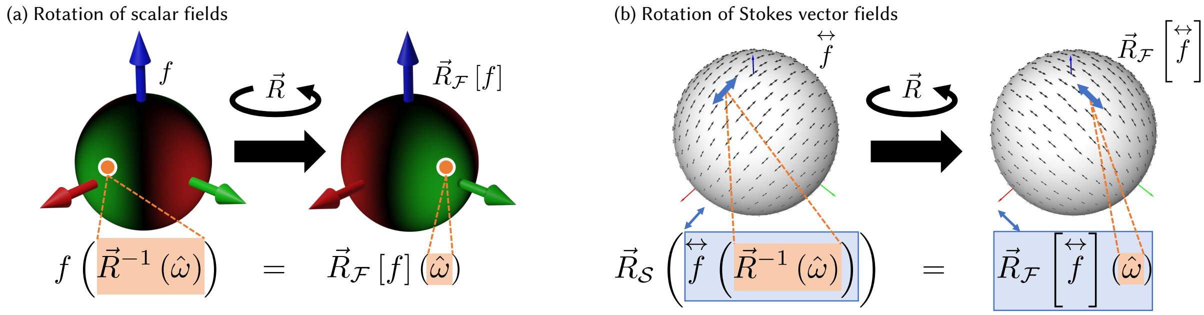

Fig. 3. Given a rotation transform $\vec{R} \in \overrightarrow{SO}(3)$, (a) a rotation of a spherical function $f : \hat{\mathbb{S}}^2 \to \mathbb{R}$ by $\vec{R}$ can be naturally defined by considering functions as textured spherical objects, which yields Equation (44). (b) In later Section 4.5, (b) We can similarly define a rotation of a Stokes vector field $\overset{\leftrightarrow}{f} : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}$ by considering it as a spherical object attached with two-sided arrows on their surface points, which is represented by Equation (97). To distinguish from the original rotation transform $\vec{R}$, which is defined as a function from single vectors to single vectors, we denote the induced rotation from functions to functions by $\vec{R}_{\mathcal{F}}$.

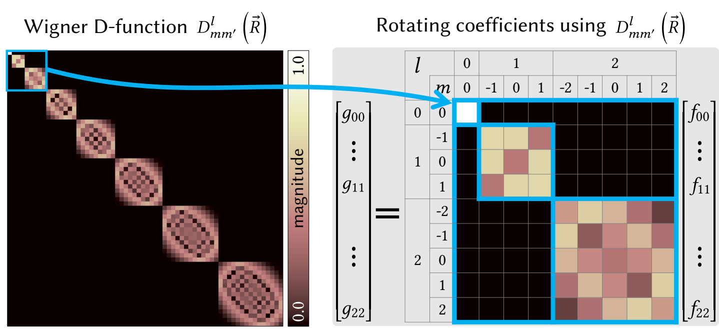

Fig. 4. Visualization of the Wigner D-function of given such $\vec{R}$ and rotating the SH coefficients. Note that the elements of the Wigner D-function are complex numbers. Thus, we visualize the matrix element by its magnitude. The matrix values are 0 when $l \neq l'$ (block-diagonal behavior) due to the Kronecker delta term, which yields the rotation invariance. The SH coefficients are rotated by simply multiplying the corresponding Wigner D-function as a coefficient matrix to the original SH coefficients without loss of information.

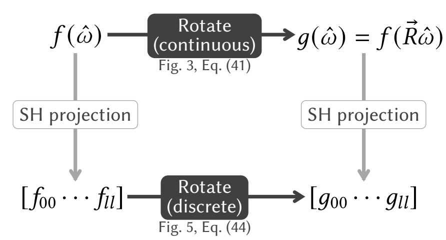

Fig. 5. The illustrative description of rotation invariance in spherical harmonics The upper path in the figure (rotate ⟶ SH projection) should be identical to the bottom path in this figure (SH projection ⟶ rotate). For rotating the discrete SH coefficients.

Proposition 2.4: Rotation invariance of spherical harmonics

Spherical harmonics $\{Y_{lm} \mid (l,m) \in I_{\mathrm{SH}}\}$, a basis of $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$, is invariant under a linear operator $\vec{R}_{\mathcal{F}}$ for any rotation $\vec{R} \in \overrightarrow{SO}(3)$ with a partition of index set $\left\{I_{\mathrm{SH},l} = \{(l',m) \in I_{\mathrm{SH}} \mid l' = l\}\right\}$. In other words, the coefficient of the linear operator $\vec{R}_{\mathcal{F}}$ with respect to spherical harmonics can be written as:

$$\left\langle Y_{lm}, \vec{R}_{\mathcal{F}}\left[Y_{l'm'}\right]\right\rangle_{\mathcal{F}\left(\hat{\mathbb{S}}^2,\mathbb{C}\right)} = 0, \quad \text{whenever } l \neq l'. \tag{45}$$

Proof: We will not cover a symbolic integration-based proof here, but there is a simple way to understand this invariance in a few steps. First, each SH function is an eigenfunction of the Laplace-Beltrami operator on $\hat{\mathbb{S}}^2$ corresponding to an eigenvalue $-l\,(l+1)$, which does not depend on m. Then the subspace of spherical functions spanned by $\{Y_{lm} \mid m \in \mathbb{Z} \text{ with } |m| \le l\}$ for fixed l is a degenerated eigenspace. Since the Laplace-Beltrami operator commutes with any rotation, the eigenspace is invariant under rotation. □

For the actual computation of rotation in the SH coefficient space, we need to know the non-zeros inner product value in the left-hand side of Equation (45) in the case of $l = l'$. This is an important special function called a *Wigner D-function*[8], which is also common in mathematics and physics. It is defined as follows:

Definition 2.5: Wigner D-function

For indices $l, m, m' \in \mathbb{Z}$ with $|m| \le l$ and $|m'| \le l$, *Wigner D-function* $D^l_{mm'} : \overrightarrow{SO}(3) \to \mathbb{C}$ is defined as follows:

$$D^l_{mm'}\left(\vec{R}\right) = \left\langle Y_{lm}, \vec{R}_{\mathcal{F}}\left[Y_{lm'}\right]\right\rangle_{\mathcal{F}\left(\hat{\mathbb{S}}^2,\mathbb{C}\right)}. \tag{46}$$

Combining Equations (45) and (46) with the Kronecker delta notation, the coefficient of a rotation transform with respect to SH can be generally rewritten as follows:

$$\left\langle Y_{lm}, \vec{R}_{\mathcal{F}}\left[Y_{l'm'}\right]\right\rangle_{\mathcal{F}\left(\hat{\mathbb{S}}^2,\mathbb{C}\right)} = \delta_{ll'} D^l_{mm'}\left(\vec{R}\right). \tag{47}$$

The coefficient matrix of Equation (47) for a particular rotation transform is shown in Figure 4. The rotation invariance of SH also appears as the block diagonal constraint on the coefficient matrix, as shown in the figure. This property also implies that we can commute the SH projection of a function and a rotation without loss of information. If one wants to obtain the SH coefficients of a function $g = \vec{R}_{\mathcal{F}}[f]$, the discrete computation between Wigner D-functions and the SH coefficients of f gives the exact same result. This process is also illustrated in Figure 5. Note that for finite coefficients up to $l \le l_{\max}$, the block diagonal sparsity produces at most $(l_{\max}+1)(2l_{\max}+1)(2l_{\max}+3)/3 = O\left(l^3_{\max}\right)$ nonzero elements.

2.3.1 Properties of Wigner D-functions. Following the definition, exact formulae for the first few Wigner D-functions are obtained as following equations using ZYZ Euler angle parameterization $D^l_{mm'}(\alpha,\beta,\gamma) := D^l_{mm'}\left(\vec{R}_{\hat{z}_g\hat{y}_g\hat{z}_g}(\alpha,\beta,\gamma)\right)$.

$$D^0_{00}(\alpha,\beta,\gamma) = 1, \qquad D^1_{mm'}(\alpha,\beta,\gamma) = \begin{array}{c|ccc} m & m'=1 & m'=0 & m'=-1 \\ \hline 1 & \frac{1+\cos\beta}{2}e^{-i(\alpha+\gamma)} & -\frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} & \frac{1-\cos\beta}{2}e^{-i(\alpha-\gamma)} \\ 0 & \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} \\ -1 & \frac{1-\cos\beta}{2}e^{i(\alpha-\gamma)} & \frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} & \frac{1+\cos\beta}{2}e^{i(\alpha+\gamma)} \end{array} \tag{48}$$

[8] Alternatively, it is known as *Wigner D-matrix* in other literature. Terminology *matrix* comes from viewing m and m' in $D^l_{mm'}$ as row and column indices of a matrix.

As seen in examples of Wigner D-functions in the above, α and γ dependencies of them can be separated as the following equation.

$$D^l_{mm'}(\alpha,\beta,\gamma) = e^{-im\alpha} d^l_{mm'}(\beta)\, e^{-im'\gamma}. \tag{49}$$

It can be directly derived from the integral (inner product) in Equation (46) by separating θ and ϕ dependencies of SH using Equation (31a). Note that the remaining β dependency, denoted by $d^l_{mm'}(\beta)$, is called a Wigner (small) d-function, but we do not need such complicated recurrence relations for it.

Additionally, note the following identities for Wigner D-functions.

Proposition 2.6: Wigner D-function indentities

(1) $D^l_{mm'}\left(\vec{I}\right) = \delta_{mm'}$, where $\vec{I} \in \overrightarrow{SO}(3)$ denotes the identity rotation.

(2) $\sum_{m_2=-l}^{l} D^l_{m_1m_2}\left(\vec{R}_1\right) D_{m_2m_3}\left(\vec{R}_2\right) = D^l_{m_1m_3}\left(\vec{R}_1\vec{R}_2\right)$

(3) $\sum_{m_2=-l}^{l} D^l_{m_1m_2}\left(\vec{R}\right) D_{m_2m_3}\left(\vec{R}^{-1}\right) = \delta_{m_1m_3}$

(4) $D^l_{mm'}\left(\vec{R}^{-1}\right) = D^l_{m'm}\left(\vec{R}\right)^*$

(5) $D^l_{-m,-m'}\left(\vec{R}\right) = (-1)^{m+m'} D^l_{mm'}\left(\vec{R}\right)^*$

(6) $D^l_{m0}\left(\vec{R}_{z_g y_g z_g}(\phi,\theta,\psi)\right) = \sqrt{\frac{4\pi}{2l+1}} Y^*_{lm}(\theta,\phi) = (-1)^m \sqrt{\frac{4\pi}{2l+1}} Y_{l,-m}(\theta,\phi)$

(7) $\left\{D^l_{mm'} \mid |m|, |m'| \le l\right\}$ is an orthogonal basis on $\mathcal{F}\left(\overrightarrow{SO}(3), \mathbb{C}\right)$, especially:

$$\left\langle D^{l_1}_{m_1m_1'}, D^{l_2}_{m_2m_2'}\right\rangle_{\mathcal{F}\left(\overrightarrow{SO}(3),\mathbb{C}\right)} = \int_{\overrightarrow{SO}(3)} D^{l_1}_{m_1m_1'}\left(\vec{R}\right)^* D^{l_2}_{m_2m_2'}\left(\vec{R}\right) \mathrm{d}\vec{R} = \frac{8\pi^2}{2l_1+1}\delta_{l_1l_2}\delta_{m_1m_2}\delta_{m_1'm_2'} \tag{50}$$

Proof: (1)—(4): Straightforward from Proposition 1.5 (1)—(3) and (5), respectively.
(5):

$$D^l_{-m,-m'}\left(\vec{R}\right) \underset{\text{Def. 2.5}}{\underset{\uparrow}{=}} \left\langle Y_{l,-m}, \vec{R}_{\mathcal{F}}\left[Y_{l,-m'}\right]\right\rangle \underset{\text{Eq. (36)}}{\underset{\uparrow}{=}} \left\langle (-1)^m Y^*_{lm}, \vec{R}_{\mathcal{F}}\left[(-1)^{m'} Y^*_{lm'}\right]\right\rangle = (-1)^{m+m'}\left\langle Y^*_{lm}, \vec{R}_{\mathcal{F}}\left[Y^*_{lm'}\right]\right\rangle.$$

Here, we observe $\left\langle Y^*_{lm}, \vec{R}_{\mathcal{F}}\left[Y^*_{lm'}\right]\right\rangle = \left\langle Y^*_{lm}, \left(\vec{R}_{\mathcal{F}}\left[Y_{lm'}\right]\right)^*\right\rangle = \left\langle Y_{lm}, \vec{R}_{\mathcal{F}}\left[Y_{lm'}\right]\right\rangle^*$. Now, we finally get the given equation.
(6) and (7): We refer to a book [Edmonds 1996]. Note that Equations (2.5.17) on p.23 and (2.5.29) on p.24 in the textbook provide an equivalent definition of SH to ours in Proposition 2.1. Equation (4.1.10) on p.55 in the book also provides the equivalent definition of Wigner D-functions to ours in Definition 2.5. Then, we can find that our propositions (6) and (7) are shown in Equations (4.1.25) on p.59 and (4.6.1) on p.62 in the book, respectively. □

2.4 Complex and Real Spherical Harmonics

Spherical harmonics defined in Equation (31a) are complex functions spaning complex-valued functions $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ with complex coefficients. However, radiometric intensity in the real world only consists of real numbers, so *real spherical harmonics*, defined as follows, sometimes makes computational efficiency.

Definition 2.7: Real spherical harmonics

$$Y_{lm}^R = \begin{cases} \sqrt{2}\Re Y_{lm}^C = \frac{1}{\sqrt{2}}\left(Y_{lm}^C + (-1)^m Y_{l,-m}^C\right) & m > 0 \\ Y_{lm}^C & m = 0 \\ \sqrt{2}\Im Y_{l|m|}^C = \frac{i}{\sqrt{2}}\left((-1)^m Y_{lm}^C - Y_{l,-m}^C\right) & m < 0 \end{cases}. \tag{51}$$

Here, Y_{lm}^C is just equal to Y_{lm} defined in Equation (31a), and we will sometimes call it *complex* spherical harmonics when we need to distinguish them from *real* ones. Note that the real spherical harmonics are also an orthonormal basis for spherical functions and have rotation invariance, but they always produce real-valued functions whenever real coefficients are given. Due to the efficiency of representing real-valued functions, most of the existing computer graphics works have used real spherical harmonics, and we also use it for some parts of polarization. However, we should know both real and complex spherical harmonics since spin-2 spherical harmonics, which will be introduced in a later section, are related to the complex ones.

The relation between complex and real spherical harmonics can be rewritten shortly by introducing a symbol $M_{m_1 m_2}^{C\to R}$ defined as:

$$\textbf{Mat}\left[M_{m_1 m_2}^{C\to R} \mid m_1, m_2 = +|m|, -|m|\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & (-1)^m \\ -i & (-1)^m i \end{bmatrix}, \text{for } |m| \neq 0, \tag{52}$$

, $M_{00}^{C\to R} = 1$, and $M_{m_1 m_2}^{C\to R} = 0$ if $|m_1| \neq |m_2|$. Similarly, a symbol $M_{m_1 m_2}^{R\to C}$ is defined as follows:

$$\textbf{Mat}\left[M_{m_1 m_2}^{R\to C} \mid m_1, m_2 = +|m|, -|m|\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ (-1)^m & -(-1)^m i \end{bmatrix}, \text{ for } |m| \neq 0. \tag{53}$$

Note that Equations (52) and (53) are unitary matrices which are the inverse of each other, and it can be written as:

$$M_{mm'}^{R\to C} = \left(M_{m'm}^{C\to R}\right)^*, \qquad \sum_{m' \in \{\pm m\}} M_{mm'}^{R\to C} M_{m'm''}^{C\to R} = \delta_{mm''}. \tag{54}$$

Here, we are using the summation symbol with $\sum_{m'\in\{\pm m\}}$ rather than much common $\sum_{m'=\pm m}$ to clarify $\sum_{m'\in\{\pm 0\}} f(m') = f(0)$ rather than $f(0) + f(0)$. Now Equation (51) can be rewritten as follows:

$$Y_{lm}^R = \sum_{m' \in \{\pm m\}} M_{mm'}^{C\to R} Y_{lm'}^C, \qquad Y_{lm}^C = \sum_{m' \in \{\pm m\}} M_{mm'}^{R\to C} Y_{lm'}^R. \tag{55}$$

On the other hand, converting coefficients of a spherical function with respect to complex real SH requires an extra complex conjugation. Suppose that $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ is a spherical function, and f_{lm}^C and f_{lm}^R are coefficients of f with respect to Y_{lm}^C and Y_{lm}^R, respectively. The following relation is obtained by the definition of SH coefficients and Equation (55):

$$\mathrm{f}_{lm}^R = \sum_{m' \in \{\pm m\}} \left(M_{mm'}^{C\to R}\right)^* \mathrm{f}_{lm'}^C, \quad \mathrm{f}_{lm}^C = \sum_{m' \in \{\pm m\}} \left(M_{mm'}^{R\to C}\right)^* \mathrm{f}_{lm'}^R. \tag{56}$$

Complex and real SH coefficients for linear operators. Similarly, we can also obtain the relation between the coefficients of a linear operator with respect to complex and real SH. Denoting a linear operator on spherical functions by $T : \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)\right)$, its $(l_o, m_o) - (l_i, m_i)$-th complex and real SH coefficients by $\mathrm{T}_{l_o m_o, l_i m_i}^C$

and $\mathrm{T}^{R}_{l_o m_o, l_i m_i}$, respectively, the following holds.

$$\begin{aligned}\mathrm{T}^{R}_{l_o m_o, l_i m_i} &= \left\langle Y^{R}_{l_o m_o}, T\left[Y^{R}_{l_i m_i}\right]\right\rangle = \left\langle \sum_{m\in\{\pm m_o\}} M^{C\to R}_{m_o m} Y^{C}_{l_o m}, T\left[\sum_{m'\in\{\pm m_i\}} M^{C\to R}_{m_i m'} Y^{C}_{l_i m'}\right]\right\rangle \\ &= \sum_{m\in\{\pm m_o\}} \sum_{m'\in\{\pm m_i\}} \left(M^{C\to R}_{m_o m}\right)^{*} M^{C\to R}_{m_i m'} \mathrm{T}^{C}_{l_o m, l_i m'}.\end{aligned} \tag{57}$$

Conversely, the following also holds.

$$\mathrm{T}^{C}_{l_o m_o, l_i m_i} = \sum_{m\in\{\pm m_o\}} \sum_{m'\in\{\pm m_i\}} \left(M^{R\to C}_{m_o m}\right)^{*} M^{R\to C}_{m_i m'} \mathrm{T}^{R}_{l_o m, l_i m'}. \tag{58}$$

Real Wigner-D functions. Similar to Equation (9) in the main paper and Definition 2.5 in this document, we can also define rotation transform for real spherical harmonics, which we call *real Wigner-D functions*, as follows:

$$D^{l,R}_{mm'}\left(\vec{R}\right) = \left\langle Y^{R}_{lm}, \vec{R}_{\mathcal{F}}\left[Y^{R}_{lm'}\right]\right\rangle. \tag{59}$$

Relation between real and complex Wigner-D functions is just a special case of Equations (57) and (58) is found by the relation between real and complex SH.

$$\begin{aligned}D^{l,R}_{mm'}\left(\vec{R}\right) &= \sum_{m_c\in\{\pm m\}} \sum_{m'_c\in\{\pm m'\}} \left(M^{C\to R}_{m m_c}\right)^{*} M^{C\to R}_{m' m'_c} D^{l,C}_{m_c m'_c}\left(\vec{R}\right), \\ D^{l,C}_{mm'}\left(\vec{R}\right) &= \sum_{m_r\in\{\pm m\}} \sum_{m'_r\in\{\pm m'\}} \left(M^{R\to C}_{m m_r}\right)^{*} M^{R\to C}_{m' m'_r} D^{l,R}_{m_r m'_r}\left(\vec{R}\right).\end{aligned} \tag{60}$$

Using this result, the relation between real Wigner-D functions and real SH (real SH version of Proposition 2.6 (6)) comes from the relation between complex ones:

$$\begin{aligned}D^{l,R}_{m0}\left(\vec{R}_{z_g y_g z_z}(\alpha,\beta,\gamma)\right) &= \sum_{m_c\in\{\pm m\}} \left(M^{C\to R}_{m m_c}\right)^{*} D^{l,C}_{m_c 0}\left(\vec{R}_{z_g y_g z_z}(\alpha,\beta,\gamma)\right) = \sqrt{\frac{4\pi}{2l+1}} \sum_{m_c\in\{\pm m\}} \left(M^{C\to R}_{m m_c}\right)^{*} Y^{C,*}_{l m_c}(\beta,\alpha) \\ &= \sqrt{\frac{4\pi}{2l+1}} Y^{R,*}_{lm}(\beta,\alpha) = \sqrt{\frac{4\pi}{2l+1}} Y^{R}_{lm}(\beta,\alpha).\end{aligned} \tag{61}$$

2.5 Azimuthally Symmetric Operators (Isotropic BRDFs)

While a general linear operator can be represented by its SH coefficients, it requires too many numbers, $(l_{\max}+1)^4$ for the maximum order $l_{\max}$, of coefficients. Several symmetry conditions for such an operator yield linear constraints on its SH coefficients, so we obtain much fewer degrees of freedom for the coefficients.

One of the common constraints of linear operators on spherical functions is azimuthal symmetry. It is defined as follows.

Definition 2.8: Azimuthally symmetric operators

Suppose that a global frame $\vec{\mathbf{F}}_g = \begin{bmatrix}\hat{x}_g & \hat{y}_g & \hat{z}_g\end{bmatrix}$ is fixed. Then a linear operator $K\colon \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)\right)$ on scalar fields is called to be *azimuthally symmetric* if it commutes with any rotation along $\hat{z}_g$, i.e.:

$$\vec{R}_{\hat{z}_g}(\alpha)_{\mathcal{F}}[K[f]] = K\left[\vec{R}_{\hat{z}_g}(\alpha)_{\mathcal{F}}[f]\right], \quad \forall\alpha \in \mathbb{R},\ \forall f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right). \tag{62}$$

When the given linear operator indicates surface interaction due to a BRDF in the rendering context, then this constraint is equivalent to the isotropy of BRDF. Suppose that the operator K has a kernel $k : \hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2 \longrightarrow \mathbb{C}$, (again, cosine-weighted BRDF in a rendering context), then the azimuthal symmetry defined in Definition 2.8 is equivalent to the following condition:

$$k(\hat{\omega}_i, \hat{\omega}_o) = k\left(\vec{R}_{\hat{z}_g}(\alpha)\,\hat{\omega}_i, \vec{R}_{\hat{z}_g}(\alpha)\,\hat{\omega}_o\right), \quad \forall\alpha \in \mathbb{R}, \quad \forall\hat{\omega}_i, \hat{\omega}_o \in \hat{\mathbb{S}}^2. \tag{63}$$

In the spherical coordinates, using the relation $\vec{R}_{\hat{z}_g}\hat{\omega}_{\text{sph}}(\theta, \phi) = \hat{\omega}_{\text{sph}}(\theta, \phi + \alpha)$ and substituting $\alpha = -\phi_i$ the above equation can be rewritten in more familiar form in computer graphics as

$$k(\theta_i, \phi_i, \theta_o, \phi_o) = k(\theta_i, 0, \theta_o, \phi_o - \phi_i). \tag{64}$$

Now, we investigate how the symmetry condition makes a linear constraint on SH coefficients.

Proposition 2.9: Coefficients of azimuthally symmetric operators (isotropic BRDFs)

Suppose that $K\colon \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)\right)$ is an azimuthally symmetric operator and $\mathrm{K}_{l_o m_o, l_i m_i} := \left\langle Y_{l_o m_o}, K\left[Y_{l_i m_i}\right]\right\rangle$ denotes the $(l_o, m_o) - (l_i, m_i)$-th coefficient of K with respect to complex SH. Then the coefficient vanishes whenever $m_i \neq m_o$, so that it can be denoted by a coefficient $\mathrm{K}_{l_o l_i m}$ with three indices such that:

$$\mathrm{K}_{l_o m_o, l_i m_i} = \delta_{m_o m_i}\mathrm{K}_{l_o l_i m}. \tag{65}$$

Proof: Start from Equation (62). First, the equation holds for any function f so that it can be rewritten as an equality of two operators. Then, taking $(l_o, m_o) - (l_i, m_i)$-th SH coefficients for both hand sides of them followed by applying Proposition 1.5 (2) yields:

$$\begin{aligned} \sum_{(l,m)\in I_{\text{SH}}} \left\langle Y_{l_o m_o}, \vec{R}_{\hat{z}_g}(\theta)_{\mathcal{F}}[Y_{lm}]\right\rangle \left\langle Y_{lm}, K\left[Y_{l_i m_i}\right]\right\rangle &= \sum_{(l,m)\in I_{\text{SH}}} \left\langle Y_{l_o m_o}, K[Y_{lm}]\right\rangle \left\langle Y_{lm}, R_{\hat{z}_g}(\theta)_{\mathcal{F}}\left[Y_{l_i m_i}\right]\right\rangle, \\ \Rightarrow \quad \sum_{(l,m)\in I_{\text{SH}}} \delta_{l_o l} D^{l_o}_{m_o m}\left(\vec{R}_{\hat{z}_g}(\theta)\right)\mathrm{K}_{lml_i m_i} &= \sum_{(l,m)\in I_{\text{SH}}} \mathrm{K}_{l_o m_o lm}\delta_{ll_i} D^{l_i}_{mm_i}\left(\vec{R}_{\hat{z}_g}(\theta)\right). \end{aligned} \tag{66}$$

From definition of Wigner D-functions in Equation (46) we easily get $D^l_{mm'}\left(\vec{R}_{\hat{g}}(\theta)\right) = \delta_{mm'}e^{-im\theta}$. Using it makes the above equation as follows:

$$\begin{aligned}\sum_{(l,m)\in I_{\mathrm{SH}}} \delta_{l_o l}\delta_{m_o m} e^{-im_o\theta}\mathrm{K}_{lml_im_i} &= \sum_{(l,m)\in I_{\mathrm{SH}}} \delta_{ll_i}\delta_{mm_i} e^{-im_i\theta}\mathrm{K}_{l_om_olm}, \\ \Rightarrow e^{-im_o\theta}\mathrm{K}_{l_om_ol_im_i} &= e^{-im_i\theta}\mathrm{K}_{l_om_ol_im_i}, \\ \Rightarrow \left(e^{-im_i\theta} - e^{-im_o\theta}\right)\mathrm{K}_{l_om_ol_im_i} &= 0.\end{aligned} \tag{67}$$

Here, we observe that $\mathrm{K}_{l_om_ol_im_i}$ should be zero for $m_i \neq m_o$ to make the above equation hold for all θ. □

Note that this property is used in Ramamoorthi and Hanrahan [2001b]. From the sparsity in Equation (65), the finite SH coefficient matrix of an azimuthally symmetric operator up to $l_i, l_o \leq l_{\max}$ has $(l_{\max}+1)\left(2l_{\max}^2 + 4l_{\max} + 3\right)/3 = O\left(l_{\max}^3\right)$ nonzero elements.

Real-SH coefficients satisfy slightly different constraints, but their constraints also have the same degree of freedom as complex ones.

Proposition 2.10: Real-SH coefficients of azimuthally symmetric operators

Suppose that $K\colon \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2,\mathbb{C}\right),\mathcal{F}\left(\hat{\mathbb{S}}^2,\mathbb{C}\right)\right)$ is an azimuthally symmetric operator and $\mathrm{K}^R_{l_om_ol_im_i} := \left\langle Y^R_{l_om_o}, K\left[Y^R_{l_im_i}\right]\right\rangle$ denotes the $(l_o,m_o)-(l_i,m_i)$-th coefficient of K with respect to real SH. Then, the coefficient satisfies the following constraints for $m \neq 0$.

$$\mathrm{K}^R_{l_om_ol_im_i} = 0 \quad \text{whenever} \quad |m_o| \neq |m_i|, \tag{68}$$

$$\mathrm{K}^R_{l_om,l_im} = \mathrm{K}^R_{l_o,-m,l_i,-m}, \quad \text{and} \quad \mathrm{K}^R_{l_om,l_i,-m} = -\mathrm{K}^R_{l_o,-m,l_im}. \tag{69}$$

Proof: Since $\mathrm{K}^R_{l_om,l_im'}$ is a linear combination of $\mathrm{K}^C_{l_o,\pm m,l_i,\pm m'}$ (four combinations of ± signs), where $\mathrm{K}^C_{mm'}$ is the $(l_o,m)-(l_i,m')$-th coefficient of K with respect to complex SH, we get Equation (68) from Proposition 2.9. Then we only have to check constraints on $\mathrm{K}_{l_o,\pm m,l_i,\pm m}$ (four combinations). Note that $\mathrm{K}^R_{l_o0,l_i0} = \mathrm{K}^C_{l_o0,l_i0}$, we should only care about cases of $m \neq 0$. Without loss of generality, suppose that $m > 0$. Rewriting Equation (57) in a matrix product with the constraint in Proposition 2.9, we get:

$$\begin{aligned}\begin{bmatrix}\mathrm{K}^R_{l_o,+m,l_i,+m} & \mathrm{K}^R_{l_o,+m,l_i,-m} \\ \mathrm{K}^R_{l_o,-m,l_i,+m} & \mathrm{K}^R_{l_o,+m,l_i,-m}\end{bmatrix} &= \begin{bmatrix}M^{C\to R}_{+m,+m} & M^{C\to R}_{+m,-m} \\ M^{C\to R}_{-m,+m} & M^{C\to R}_{-m,-m}\end{bmatrix}^* \begin{bmatrix}\mathrm{K}^C_{l_o,+m,l_i,+m} & 0 \\ 0 & \mathrm{K}^C_{l_o,-m,l_i,-m}\end{bmatrix} \begin{bmatrix}M^{R\to C}_{+m,+m} & M^{R\to C}_{+m,-m} \\ M^{R\to C}_{-m,+m} & M^{R\to C}_{-m,-m}\end{bmatrix}^* \\ &= \frac{1}{2}\begin{bmatrix}1 & (-1)^m \\ i & -(-1)^m i\end{bmatrix}\begin{bmatrix}\mathrm{K}^C_{l_o,+m,l_i,+m} & 0 \\ 0 & \mathrm{K}^C_{l_o,-m,l_i,-m}\end{bmatrix}\begin{bmatrix}1 & -i \\ (-1)^m & (-1)^m i\end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix}\mathrm{K}^C_{l_o,+m,l_i,+m} + \mathrm{K}^C_{l_o,-m,l_i,-m} & -i\left(\mathrm{K}^C_{l_o,+m,l_i,+m} - \mathrm{K}^C_{l_o,-m,l_i,-m}\right) \\ i\left(\mathrm{K}^C_{l_o,+m,l_i,+m} - \mathrm{K}^C_{l_o,-m,l_i,-m}\right) & \mathrm{K}^C_{l_o,+m,l_i,+m} + \mathrm{K}^C_{l_o,-m,l_i,-m}\end{bmatrix}.\end{aligned}$$

The right-hand side implies Equation (69). □

2.6 Spherical Convolution

2.6.1 Senses to define convolution. Before investigating spherical convolution, let's review about convolution on planar (Euclidean) domains, $\mathbb{R}^n$. First, the convolution of two functions k and $f \in \mathcal{F}(\mathbb{R}^n, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is defined by $k * f(\mathbf{x}) = \int_{\mathbb{R}^n} k(\mathbf{x} - \mathbf{x}') f(\mathbf{x}') \, \mathrm{d}\mathbf{x}'$. While it is a binary operation of functions in $\mathcal{F}(\mathbb{R}^n, \mathbb{C})$ into the same function space $\mathcal{F}(\mathbb{R}^n, \mathbb{C})$ yet, this property no more holds for spherical domains. To extend the definition of convolution to spherical domains, consider a linear operator $K \in \mathcal{L}(\mathcal{F}(\mathbb{R}^n, \mathbb{C}), \mathcal{F}(\mathbb{R}^n, \mathbb{C}))$ defined by $K[f] = k * f$. Then we observe an important property that K is a translation equivariant linear operator, i.e., it commutes an arbitrary translation. Conversely, if a translation equivariant linear operator K is given first, then there exists some function $k \in \mathcal{F}(\mathbb{R}^n, \mathbb{C})$ such that $K[f] = k * f$ under the assumption of the existence of an operator kernel of K.

2.6.2 Spherical convolution. Spherical convolution is defined as a binary operation of an azimuthally symmetric spherical function $k : [0, \pi] \to \mathbb{R}$ and a spherical function $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{R}\right)$ that does not need to have any symmetry. Note that azimuthal symmetry of spherical functions, not operators, is discussed in Section 2.1.1.

Definition 2.11: Spherical convolution

k and $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are spherical functions. Suppose that k has azimuthal symmetry. Then spherical convolution of k and f is defined as follows:

$$k * f(\hat{\omega}) = \int_{\hat{\mathbb{S}}^2} k\left(\cos^{-1} \hat{\omega} \cdot \hat{\omega}'\right) f(\hat{\omega}') \, \mathrm{d}\hat{\omega}'. \tag{70}$$

In this operation, k is called the *convolution kernel*.

Due to the azimuthal symmetry of k, Equation (70) can be rewritten in several forms using the following property:

$$k\left(\cos^{-1} \hat{\omega} \cdot \hat{\omega}'\right) = k\left(\vec{R}_{\hat{\omega} \to \hat{z}_g} \hat{\omega}'\right) = k\left(\vec{R}_{\hat{\omega}' \to \hat{z}_g} \hat{\omega}\right), \tag{71}$$

Proof: Recall that the inner product is preserved under rotation, as written in Equation (10). Then we get

$$k\left(\cos^{-1} \hat{\omega} \cdot \hat{\omega}'\right) \underset{\substack{\uparrow \\ \text{Eq. (10)}}}{=} k\left(\cos^{-1}\left(\vec{R}_{\hat{\omega} \to \hat{z}_g} \hat{\omega}\right) \cdot \left(\vec{R}_{\hat{\omega} \to \hat{z}_g} \hat{\omega}'\right)\right) = k\left(\cos^{-1} \hat{z}_g \cdot \left(\vec{R}_{\hat{\omega} \to \hat{z}_g} \hat{\omega}'\right)\right)$$
$$\underset{\substack{\uparrow \\ \text{Eq. (38a)}}}{=} k\left(\vec{R}_{\hat{\omega} \to \hat{z}_g} \hat{\omega}'\right).$$

Then the remaining term can also be obtained in the same way. □

where in the first term, k is written as a function of a single real value of zenith angle, and in the second and third terms, $\vec{R}_{\hat{a} \to \hat{b}}$ denotes any rotation in $\overrightarrow{SO}(3)$ such that $\vec{R}_{\hat{a} \to \hat{b}} \hat{a} = \hat{b}$. Note that the second and third terms are well-defined independent of choices of such rotations due to the symmetry of k. Note that we can rewrite

Equation (70) in two other forms as follows:

$$k * f = \int_{\hat{\mathbb{S}}^2} f(\hat{\omega}') \vec{R}_{\hat{z}_g \to \hat{\omega}', \mathcal{F}}[k] \, d\hat{\omega}', \tag{72}$$

$$k * f(\hat{\omega}) = \left\langle \vec{R}_{\hat{z}_g \to \hat{\omega}, \mathcal{F}}[k^*], f \right\rangle_{\mathcal{F}}. \tag{73}$$

While Equation (72) views the convolution as a linear combination of rotated kernel, Equation (73) views a single point at the operation result as an inner product of the kernel k and the operand function f. When approximating such integral operations on a discrete point set of the domain $\hat{\mathbb{S}}^2$, we can consider each function k and f as numeric vectors whose indices indicate each point on $\hat{\mathbb{S}}^2$, and the convolution operation can be considered as a matrix related to k. Then Equation (72) can be considered as a linear combination of column vectors of the matrix of k, while Equation (73) does as the inner product of a row vector the matrix of k and the vector of f. We call these views *column view of convolution* and *row view of convolution*, respectively. While in scalar spherical convolution, the format of the kernel k in the two views seem straightforwardly equivalent, excepting just complex conjugation, in polarized spherical convolution, which will be introduced in a later section, the kernel will be defined slightly differently depending on each view. In that section, we will focus on the column view in Equation (72), which is related to the view of convolution as a linear operator, which will be introduced now.

Rather than viewing the convolution as a binary operation on spherical functions, it can be considered as a special case of linear operation on spherical functions with fixing the kernel. The following key property of spherical convolution as a linear operator explains why spherical convolution is defined in the above way.

Proposition 2.12: Spherical convolution and rotation equivariance

Suppose that a linear operator $K_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)\right)$ on spherical functions is rotation equivariant, i.e., $K_{\mathcal{F}}\left[\vec{R}_{\mathcal{F}}[f]\right] = \vec{R}_{\mathcal{F}}[K_{\mathcal{F}}[f]]$ for any $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)$, and has an operator kernel $K \in \mathcal{F}\left(\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2, \mathbb{K}\right)$. Then the linear operator $K_{\mathcal{F}}$ is characterized by a spherical convolution with a function $k : [0, \pi] \to \mathbb{K}$, i.e., $T[f] = k * f$. Here, the kernel is obtained as:

$$k(\theta) := K(\hat{\omega}, \hat{\omega}') \text{ for any } \hat{\omega}, \hat{\omega}' \in \hat{\mathbb{S}}^2 \text{ with } \hat{\omega} \cdot \hat{\omega}' = \cos\theta. \tag{74}$$

Moreover,

$$k = K_{\mathcal{F}}\left[\delta\left(\hat{\omega}, \hat{z}_g\right)\right]. \tag{75}$$

Conversely, convolution $k * f$ is rotation equivariant for f.

Proof: For any function f,

$$K_{\mathcal{F}}[f](\hat{\omega}_0) = \int_{\hat{\mathbb{S}}^2} K(\hat{\omega}_i, \hat{\omega}_o) f(\hat{\omega}_i) \, d\hat{\omega}_i = \int_{\hat{\mathbb{S}}^2} k(\theta) f(\hat{\omega}_i) \, d\hat{\omega}_i, \tag{76}$$

where $\cos\theta = \hat{\omega}_i \cdot \hat{\omega}_o$. Then it is equivalent to Definition 2.11.
For Equation (75),

$$K_{\mathcal{F}}\left[\delta\left(\hat{\omega}, \hat{z}_g\right)\right](\hat{\omega}_o) = \int_{\hat{\mathbb{S}}^2} K(\hat{\omega}_i, \hat{\omega}_o) \, \delta\left(\hat{\omega}_i, \hat{z}_g\right) d\hat{\omega}_i = K\left(\hat{z}_g, \hat{\omega}_o\right) = k(\theta_o), \tag{77}$$

where $\hat{\omega}_o = \hat{\omega}_{\text{sph}}(\theta_o, \phi_i)$. □

Here, it is worth noting not to confuse the operator kernel and the convolution kernel. The rotation equivariant linear operator $K_{\mathcal{F}}$ is characterized by the *operator kernel* K, and at the same time by the *convolution kernel* k, where $K(\hat{\omega}, \hat{\omega}') = k\left(\cos^{-1} \hat{\omega} \cdot \hat{\omega}'\right)$ holds. When handling with rotation equivariant operators, some formulae require distinction of the two types of kernels.

2.6.3 Convolution in spherical harmonics. As Fourier transform (both continuous and discrete versions) reduces convolution into the simpler pointwise product in the Euclidean domain, spherical harmonics can reduce the integral formula of spherical convolution in Equation (70) into the following formula for coefficient vectors, which is almost an element-wise product.

Proposition 2.13: Spherical convolution theorem: convolution in SH coefficients

Denote SH coefficients of an azimuthally symmetric spherical function $k \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)$ by k_{l0} and SH coefficients of a spherical function $f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)$ by $\mathrm{f}_{lm} \coloneqq \langle Y_{lm}, f\rangle_{\mathcal{F}}$. Then, the SH coefficient of the convolution $k * f$ can be evaluated as follows:

$$\langle Y_{lm}, k * f\rangle_{\mathcal{F}} = \sqrt{\frac{4\pi}{2l+1}}\mathrm{k}_{l0}\mathrm{f}_{lm}. \tag{78}$$

Proof: We refer to [Driscoll and Healy 1994]. □

Considering convolution with a fixed kernel as a linear operator, the above fact can be rewritten in terms of a coefficient matrix.

Proposition 2.14: Spherical convolution theorem: linear operator form

A rotation equivariant linear operator $K_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)\right)$ is characterized by the convolution kernel $k \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{K}\right)$. Denote the SH coefficients of k by $\mathrm{k}_{l0} \coloneqq \langle Y_{l0}, k\rangle_{\mathcal{F}}$. Then the SH coefficients of $K_{\mathcal{F}}$, denoted by $\mathrm{K}_{l_o m_o, l_i m_i} \coloneqq \left\langle Y_{l_o m_o}, K_{\mathcal{F}}\left[Y_{l_i m_i}\right]\right\rangle_{\mathcal{F}}$, are evaluated as follows.

$$\mathrm{K}_{l_o m_o, l_i m_i} = \delta_{l_o l_i}\delta_{m_o m_i}\sqrt{\frac{4\pi}{2l+1}}\mathrm{k}_{l0}. \tag{79}$$

Imagine that the element-wise product of two vectors with a fixed left operand is equivalent to the matrix-vector product with a diagonal matrix.

In Section 5.8 later, we will derive our polarized spherical convolution theorem using the new basis as a generalization of Proposition 2.14.

2.7 Reflection operator in SH

In the context of rendering, we sometimes need a reflection operator which flips $\hat{\omega} \in \hat{\mathbb{S}}^2$ with respect to a given axis.

We call a transform $T : \hat{\mathbb{S}}^2 \rightarrow \hat{\mathbb{S}}^2$ as the reflection operator along $\hat{z}$, if

$$T\left(\vec{\mathbf{F}}\left[\omega_1, \omega_2, \omega_3\right]^T\right) = T\left(\vec{\mathbf{F}}\left[\omega_1, \omega_2, -\omega_3\right]^T\right), \tag{80}$$

for any $\hat{\omega} = \vec{\mathbf{F}}\left[\omega_1, \omega_2, \omega_3\right]^T \in \hat{\mathbb{S}}^2 \subset \vec{\mathbb{R}}^3$ and $\vec{\mathbf{F}} \in \vec{\mathbb{F}}^3$ such that $\vec{\mathbf{F}}\left[:, 3\right] = \hat{z}$. Note that it is well-defined independent of choice of the frame $\vec{\mathbf{F}}$. Note that it is self inversion and it acts on $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right)$ as a linear operator as follows:

$$T_{\mathcal{F}}\left[f\right]\left(\hat{\omega}\right) = f\left(T\left(\hat{\omega}\right)\right) = f\left(T^{-1}\left(\hat{\omega}\right)\right), \quad \forall f \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathbb{C}\right). \tag{81}$$

Then, its SH coefficients can be obtained as follows:

$$\left\langle Y_{lm}, T_{\mathcal{F}}\left[Y_{l'm'}\right]\right\rangle = \int_{\mathbb{S}^2} Y_{lm}^*\left(\theta, \phi\right) Y_{l'm'}\left(\pi - \theta, \phi\right) \mathrm{d}\omega = \delta_{ll'}\delta_{mm'}\left(-1\right)^{l+m}. \tag{82}$$

3 BACKGROUND: POLARIZATION AND MUELLER CALCULUS

Here, we introduce the theoretical background of polarization in Mueller calculus. Section 3.1 gives brief introduction for novice readers who are not familiar with Mueller calculus formulation. Section 3.2 provides a reformulation of it in a more rigorous manner to construct a solid theory of our polarized SH in later sections. Section 3.2 is aimed at dedicated readers who are familiar with rigorous mathematics. While Mueller calculus and its formal definition using equivalence classes already exist, this section contains our novel usage of terminology which distinguishes *Stokes vectors* and *Stokes component vectors* and notations $[\cdot]_{\vec{\mathbf{F}}}$ and $[\cdot]^{\vec{\mathbf{F}}}$.

3.1 Introduction to Mueller Calculus

To take polarization into account, several intensity-related quantities, including radiance and BSDF, should be reformulated. The polarized intensity of rays is usually described by Jones calculus, which includes phase information of electromagnetic waves, or Mueller calculus, which includes unpolarized intensity due to incoherent light. Following recent works in computer graphics [Baek et al. 2018, 2020; Hwang et al. 2022] we focus on Mueller calculus.

Suppose that there is a polarized ray and a local frame $\vec{\mathbf{F}} = [\hat{x}, \hat{y}, \hat{z}]$, where $\hat{z}$ is equal to the propagation direction of the ray. Then the polarized intensity of the ray is characterized by the four Stokes parameters $\mathbf{s} = [s_0, s_1, s_2, s_3]^T$. Here, each component s_0 to s_3 indicates total intensity, linear polarization in horizontal/vertical direction, linear polarization of diagonal/anti-diagonal direction, and circular polarization, respectively. We refer interested readers to Collett [2005] for a more physical foundation of polarization and Mueller calculus.

While Stokes parameters have linearity so that Stokes parameters obtained under multiple incoherent light sources are equal to the addition of Stokes parameters obtained under each individual source, they have an important property that makes them different from scalars and even vectors.

When taking another local frame $\vec{\mathbf{F}}' = \vec{R}_{\hat{z}}(\vartheta)\,\vec{\mathbf{F}}$, obtained by rotating $\vec{\mathbf{F}}$ by ϑ along its z axis, the Stokes parameters with respect to the new frame $\vec{\mathbf{F}}'$ is evaluated as

$$\mathbf{s}' = \mathbf{C}_{\vec{\mathbf{F}} \to \vec{\mathbf{F}}'}\mathbf{s} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\vartheta & \sin 2\vartheta & 0 \\ 0 & -\sin 2\vartheta & \cos 2\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{s}. \tag{83}$$

We can observe here that s_0 and s_3 behave as *scalar*s, which are measured independent of local frames. On the other hand, s_1 and s_2 are neither scalars nor coordinates of an ordinary vector, which must have ϑ rather than 2ϑ in Equation (83). This twice rotation property of s_1 and s_2 under coordinate conversion will be dealt with as *spin-2 functions* in Section 5.1. Figure 6(a) visualizes it where the two-sided arrow in the left indicates the actual oscillation direction of polarized ray and the right plot shows s_1 and s_2 values of it under a local frame. Figure 6(b) also visualizes coordinate conversion of a fixed ray.

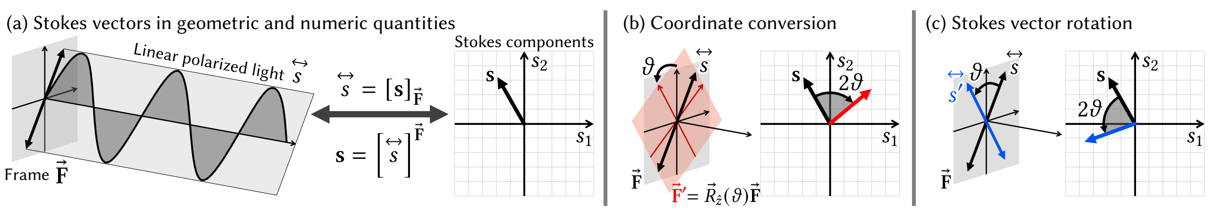

Fig. 6. (a) We distinguish a Stokes vector $\overset{\leftrightarrow}{s}$ as geometric quantities and its Stokes component vector $\mathbf{s}$ as numeric quantities. (b) Under coordinates conversion, the Stokes component vectors rotate twice while the Stokes vector $\overset{\leftrightarrow}{s}$ does not change. (c) We also define the rotation of the Stokes vector itself.

3.2 Formal Definitions of Mueller Calculus

Following Mojzík et al. [2016], the *Stokes space* and *Stokes vectors* can be formally defined using equivalence classes. Here, we distinguish *spin-2 Stokes vectors*, which consists of s_1 and s_2 linear polarization components and *full Stokes vectors* (or simply *Stokes vectors*) to build our polarized SH theory, which requires separating linear operators (Mueller matrix or transforms) into spin-0 and spin-2 parts.

Definition 3.1: Spin-2 Stokes spaces

For any $\hat{\omega} \in \hat{\mathbb{S}}^2$, the *Spin-2 Stokes space with respect to* $\hat{\omega}$, denoted by $\mathcal{S}^2_{\hat{\omega}}$ is defined as follows.

$$\mathcal{S}^2_{\hat{\omega}} := \left\{ \left[\left(\mathbf{s}, \vec{\mathbf{F}} \right) \right]_\sim \mid \mathbf{s} \in \mathbb{R}^2, \vec{\mathbf{F}} \in \vec{\mathbb{F}}^3_{\hat{\omega}} \right\} \tag{84}$$

Here, $[\cdot]_\sim$ denotes an an equivalence class with respect to a relation $\sim$ on a pair of a numeric vector in $\mathbb{R}^2$ and a frame in $\vec{\mathbb{F}}^3_{\hat{\omega}}$ defined as:

$$\left(\mathbf{s}, \vec{\mathbf{F}} \right) \sim \left(\mathbf{t}, \vec{\mathbf{G}} \right) \text{ if and only if } \mathbf{t} = \mathbf{R}_2\left(-2\vartheta\right)\mathbf{s}, \quad \forall \mathbf{s}, \mathbf{t} \in \mathbb{R}^2, \vec{\mathbf{F}}, \vec{\mathbf{G}} \in \vec{\mathbb{F}}^3_{\hat{\omega}}, \tag{85}$$

where ϑ is uniquely determined to satisfy $\vec{\mathbf{G}} = \vec{\mathbf{F}}\mathbf{R}_z\left(\vartheta\right)$ up to $+2n\pi$.

Note that our main paper writes as $\vec{\mathbf{G}} = \vec{R}_{\hat{z}}\left(\vartheta\right)\vec{\mathbf{F}}$, where $\hat{z} = \vec{\mathbf{F}}$, to avoid introducing notations for numeric rotations. These are equivalent due to a relationship discussed in Section 1.1. Now we introduce Stokes vectors, which are geometric quantities, and Stokes components, which are numeric ones, and notations to convert them to each other.

Definition 3.2: Spin-2 Stokes vectors and spin-2 Stokes component vectors

Using notations Definition 3.1, we denote $[\mathbf{s}]_{\vec{\mathbf{F}}} := \left[\left(\mathbf{s}, \vec{\mathbf{F}} \right) \right]_\sim \in \mathcal{S}^2_{\hat{\omega}}$, which called a *spin-2 Stokes vector of a ray along* $\hat{\omega}$. $\mathbf{s}$ is called the *spin-2 Stokes component vector of* $[\mathbf{s}]_{\vec{\mathbf{F}}}$ *with respect to* $\vec{\mathbf{F}}$. Conversely, for any $\overleftrightarrow{s} \in \mathcal{S}^2_{\hat{\omega}}$, $\left[\overleftrightarrow{s}\right]^{\vec{\mathbf{F}}}$ is defined as some $\mathbf{s}' \in \mathbb{R}^2$ which satisfies $\overleftrightarrow{s} = [\mathbf{s}']_{\vec{\mathbf{F}}}$. Note that it is well-defined, independent of the choice of a frame[9].

Now, full Stokes vectors can be defined similarly or just by taking the direct sum of scalars and spin-2 Stokes vectors.

[9]Our $[\cdot]_{\vec{\mathbf{F}}}$ and $[\cdot]^{\vec{\mathbf{F}}}$ notations are slightly inspired from a convention in Riemannian geometry, where coordinates v^i which depends on an observer can be converted to an invariant quantity $v^i\mathbf{e}_i$ by attaching the subscripted quantity $\mathbf{e}_i$, which indicates a basis for the local tangent space.

Definition 3.3: (Full) Stokes spaces

For any $\hat{\omega} \in \hat{\mathbb{S}}^2$, the *full Stokes space* (or *Stokes space*, simply) *with respect to* $\hat{\omega}$, denoted by $\mathcal{S}^4_{\hat{\omega}}$ (or $\mathcal{S}_{\hat{\omega}}$) is defined by two ways, equivalently.

(1) $\mathcal{S}^4_{\hat{\omega}} := \mathbb{R} \oplus \mathcal{S}^2_{\hat{\omega}} \oplus \mathbb{R}$

(2) $\mathcal{S}^4_{\hat{\omega}} := \left\{ \left[\left(\mathbf{s}, \vec{\mathbf{F}}\right)\right]_\sim \mid \mathbf{s} \in \mathbb{R}^4, \vec{\mathbf{F}} \in \vec{\mathbb{F}}^3_{\hat{\omega}} \right\}$, where $\left(\mathbf{s}, \vec{\mathbf{F}}\right) \sim \left(\mathbf{t}, \vec{\mathbf{G}}\right)$ if and only if $\mathbf{t} = \mathbf{C}_{\vec{\mathbf{F}}\to\vec{\mathbf{G}}}\mathbf{s}$.

Here, $\mathbf{C}_{\vec{\mathbf{F}}\to\vec{\mathbf{G}}}$ is defined using ϑ such that $\vec{\mathbf{G}} = \vec{\mathbf{F}}\mathbf{R}_z(\vartheta)$ as follows.

$$\mathbf{C}_{\vec{\mathbf{F}}\to\vec{\mathbf{G}}} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\vartheta & \sin 2\vartheta & 0 \\ 0 & -\sin 2\vartheta & \cos 2\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{86}$$

Here, we sometimes denote the matrix in the right-hand side of Equation (86) as $\mathbf{R}_{1:2}(-2\vartheta)$, which indicates embed $\mathbf{R}_2$ into a 4×4 matrix (with index based on zero) at indices 1 and 2.

We also define the *(entire) spin-2 Stokes space* as $\mathcal{S}^2 := \sqcup_{\hat{\omega}\in\hat{\mathbb{S}}^2}\mathcal{S}^2_{\hat{\omega}}$ and the *(entire) Stokes space* as $\mathcal{S}^4 := \sqcup_{\hat{\omega}\in\hat{\mathbb{S}}^2}\mathcal{S}^4_{\hat{\omega}}$, where $\sqcup$ indicates disjoint union[10]. (Full) Stokes vectors, (full) Stokes components, and $[\cdot]_{\vec{\mathbf{F}}}$ and $[\cdot]_{\vec{\mathbf{F}}}$ notations from Definition 3.2 can be redefined for full Stokes spaces similarly. Note that for $\overleftrightarrow{s}_2 := \left[[s_1, s_2]^T\right]_{\vec{\mathbf{F}}} \in \mathcal{S}^2_{\hat{\omega}}$ and $\overleftrightarrow{s}_4 := \left[[s_0, s_1, s_2, s_3]^T\right]_{\vec{\mathbf{F}}} \in \mathcal{S}^4_{\hat{\omega}}$, we use notations for theirs relationship as $\overleftrightarrow{s}_4 = s_0 \oplus \overleftrightarrow{s}_2 \oplus s_3$ or $\overleftrightarrow{s}_4 = \left(s_0, \overleftrightarrow{s}_2, s_3\right)$.

We define operations on (spin-2) Stokes vectors, which are well-defined independent of the choice of a frame $\vec{\mathbf{F}} \in \vec{\mathbb{F}}^3_{\hat{\omega}}$ below.

Definition 3.4: Stokes vector operations

For $\overleftrightarrow{s}$ and $\overleftrightarrow{t} \in \mathcal{S}^{\{2,4\}}_{\hat{\omega}}$,

(1) Linear combination: for any $a, b \in \mathbb{R}$, $a\overleftrightarrow{s} + a\overleftrightarrow{t} := \left[a\left[\overleftrightarrow{s}\right]^{\vec{\mathbf{F}}} + a\left[\overleftrightarrow{t}\right]^{\vec{\mathbf{F}}}\right]_{\vec{\mathbf{F}}}$ for any $\vec{\mathbf{F}} \in \vec{\mathbb{F}}^3_{\hat{\omega}}$.

(2) Inner product: $\left\langle \overleftrightarrow{s}, \overleftrightarrow{t} \right\rangle_{\mathcal{S}^{\{2,4\}}_{\hat{\omega}}} := \left[\overleftrightarrow{s}\right]^{\vec{\mathbf{F}}} \cdot \left[\overleftrightarrow{t}\right]^{\vec{\mathbf{F}}}$ (or denoted as simply $\langle \cdot, \cdot \rangle_{\mathcal{S}}$, or explicitly $\langle \cdot, \cdot \rangle_{\mathcal{S}|\mathbb{R}}$, etc.)

(3) Rotation: for any $\vec{R} \in \overrightarrow{SO}(3)$, $\vec{R}_{\mathcal{S}} \in \mathcal{L}(\mathcal{S}, \mathcal{S})$ is defined as $\vec{R}_{\mathcal{S}}\overleftrightarrow{s} = \left[\left[\overleftrightarrow{s}\right]^{\vec{\mathbf{F}}}\right]_{\vec{R}\vec{\mathbf{F}}}$.

When $\overleftrightarrow{s}$ and $\overleftrightarrow{t} \in \mathcal{S}^2_{\hat{\omega}}$, the following is additionally defined.

(1) Complex scalar multiplication: for any $z \in \mathbb{C}$, $z\overleftrightarrow{s} := \left[\mathbb{R}^2\left(z\mathbb{C}\left(\left[\overleftrightarrow{s}\right]^{\vec{\mathbf{F}}}\right)\right)\right]_{\vec{\mathbf{F}}}$

(2) Inner product over scalar $\mathbb{C}$: $\left\langle \overleftrightarrow{s}, \overleftrightarrow{t} \right\rangle_{\mathcal{S}^2_{\hat{\omega}}|\mathbb{C}} := \mathbb{C}\left(\left[\overleftrightarrow{s}\right]^{\vec{\mathbf{F}}}\right)^* \mathbb{C}\left(\left[\overleftrightarrow{t}\right]^{\vec{\mathbf{F}}}\right) \in \mathbb{C}$ (or denoted simply $\langle \cdot, \cdot \rangle_{\mathcal{S}|\mathbb{C}}$).

Here, $\mathbb{C} : \mathbb{R}^2 \to \mathbb{C}$ and $\mathbb{R}^2 : \mathbb{C} \to \mathbb{R}^2$ denote the canonical conversions between $\mathbb{R}^2$ and $\mathbb{C}$. In addition, note that we sometimes denotes $[z]_{\vec{\mathbf{F}}} := \left[[\Re z, \Im z]^T\right]_{\vec{\mathbf{F}}} \in \mathcal{S}^2_{\vec{\mathbf{F}}[:,3]}$ for a complex number $z \in \mathbb{C}$.

We observe that $\mathcal{S}^4_{\hat{\omega}}$ is an inner product space over $\mathbb{R}$, while $\mathcal{S}^2_{\hat{\omega}}$ can be handled as an inner product space over both $\mathbb{R}$ or $\mathbb{C}$. Two inner products satisfy the relationship described in Proposition 1.10.

[10]For readers who are not familiar to disjoint union, it can be just considered as union.

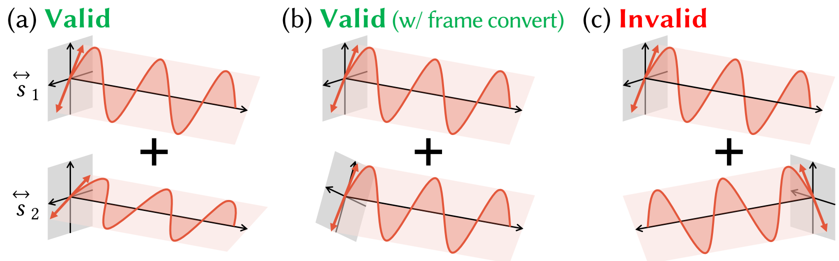

Fig. 7. Addition between two Stokes vectors $\overleftrightarrow{s}_1$ and $\overleftrightarrow{s}_2$. (a) If two Stokes component vectors have the same frame, we can perform addition directly. (b) If two Stokes component vectors have different frames but on the same Stokes space, addition can be performed with frame conversion. (c) If two Stokes vectors belong to different Stokes spaces (different ray directions), addition cannot be defined.

Not only just a vector space, linear operators (transforms) also have to be formulated in Mueller calculus.

Definition 3.5: Mueller transform space

The *(full) Mueller space with respect to $\hat{\omega}_i$ and $\hat{\omega}_o \in \hat{\mathbb{S}}^2$*, denoted by $\mathcal{M}^4_{\hat{\omega}_i\to\hat{\omega}_o}$ (or $\mathcal{M}_{\hat{\omega}_i\to\hat{\omega}_o}$ simply), and *spin 2-to-2 Mueller space with respect to $\hat{\omega}_i$ and $\hat{\omega}_o$*, denoted by $\mathcal{M}^2_{\hat{\omega}_i\to\hat{\omega}_o}$ are defined as follows, equivalently.

(1) $\mathcal{M}^{\{2,4\}}_{\hat{\omega}_i\to\hat{\omega}_o} := \mathcal{L}\left(\mathcal{S}^{\{2,4\}}_{\hat{\omega}_i}, \mathcal{S}^{\{2,4\}}_{\hat{\omega}}\right)$, respectively.

(2) $\mathcal{M}^{\{2,4\}}_{\hat{\omega}_i\to\hat{\omega}_o} := \left\{\left[\left(\mathbf{M}, \vec{\mathbf{F}}_i, \vec{\mathbf{F}}_o\right)\right]_\sim \mid \mathbf{M} \in \mathbb{R}^{\{2\times 2, 4\times 4\}}, \vec{\mathbf{F}}_i \in \vec{\mathbb{F}}^3_{\hat{\omega}}, \vec{\mathbf{F}}_o \in \vec{\mathbb{F}}^3_{\hat{\omega}_o}\right\}$, where $\left(\mathbf{M}, \vec{\mathbf{F}}_i, \vec{\mathbf{F}}_o\right) \sim \left(\mathbf{N}, \vec{\mathbf{G}}_i, \vec{\mathbf{G}}_o\right)$ if and only if

$$\begin{aligned} \mathbf{N} &= \mathbf{R}_2\left(-2\vartheta_o\right)\mathbf{M}\mathbf{R}_2\left(2\vartheta_i\right), \text{ for } \mathcal{M}^2_{\hat{\omega}_i\to\hat{\omega}_o}, \\ \mathbf{N} &= \mathbf{C}_{\vec{\mathbf{F}}_o\to\vec{\mathbf{G}}_o}\mathbf{M}\mathbf{C}^{-1}_{\vec{\mathbf{F}}_i\to\vec{\mathbf{F}}_i}, \text{ for } \mathcal{M}^4_{\hat{\omega}_i\to\hat{\omega}_o}, \end{aligned} \tag{87}$$

where $\vec{\mathbf{G}}_i = \mathbf{F}_i\mathbf{R}_z\left(\vartheta_i\right)$, $\vec{\mathbf{G}}_o = \mathbf{F}_o\mathbf{R}_z\left(\vartheta_o\right)$, and $\mathbf{C}$ from Equation (86).

Similar to Stokes spaces, we can define the *(entire) Mueller space* as $\mathcal{M}^{\{2,4\}} := \sqcup_{\hat{\omega}_i,\hat{\omega}_o\in\hat{\mathbb{S}}^2}\mathcal{M}^{\{2,4\}}_{\hat{\omega}_i\to\hat{\omega}_o}$ in both senses of spin-2 and full. As a full Stokes vector contains a spin-2 Stokes vector as its subpart, a full Mueller transform $\overleftrightarrow{M} \in \mathcal{M}^4$ contains a spin 2-to-2 Mueller transform as its subpart, which is denoted by $\overleftrightarrow{M}[1{:}2, 1{:}2] \in \mathcal{M}^2$. Note that separately taking a single index 1 or 2 for $\overleftrightarrow{M}$ is illegal since it yields a frame-dependent quantity. We also define *Mueller matrices*, numeric quantities measured from Mueller transforms.

Definition 3.6: Mueller transforms and Mueller matrices

Using notations Definition 3.5, we denote $[\mathbf{M}]_{\vec{\mathbf{F}}_i\to\vec{\mathbf{F}}_o} := \left[\left(\mathbf{M}, \vec{\mathbf{F}}_i, \vec{\mathbf{F}}_o\right)\right]_\sim \in \mathcal{M}^{\{2,4\}}_{\hat{\omega}_i\to\hat{\omega}_o}$, which called a *Mueller transform from a ray along $\hat{\omega}_i$ to one along $\hat{\omega}_o$*. $\mathbf{M}$ is called the *Mueller matrix of* $[\mathbf{M}]_{\vec{\mathbf{F}}_i\to\vec{\mathbf{F}}_o}$ *with respect to* $\vec{\mathbf{F}}_i$ *and* $\vec{\mathbf{F}}_o$. Conversely, for any $\overleftrightarrow{M} \in \mathcal{M}^{\{2,4\}}_{\hat{\omega}_i\to\hat{\omega}_o}$, $\left[\overleftrightarrow{M}\right]^{\vec{\mathbf{F}}_i\to\vec{\mathbf{F}}_o}$ is defined as some $\mathbf{M}' \in \mathbb{R}^{\{2\times 2, 4\times 4\}}$ which satisfies $\overleftrightarrow{M} = [\mathbf{M}']_{\vec{\mathbf{F}}_i\to\vec{\mathbf{F}}_o}$. Note that it is well-defined, independent of the choice of frames.

Since a Mueller space is a space of linear maps, linear combination and product between two Mueller transforms in the same space is naturally defined. For a rotation $\vec{R} \in \overrightarrow{SO}(3)$, $\vec{R}_{\mathcal{M}} : \mathcal{L}(\mathcal{M}, \mathcal{M})$ is defined as:

$$\vec{R}_{\mathcal{M}}\left[\overleftrightarrow{M}\right] = \vec{R}_{\mathcal{S}}\overleftrightarrow{M}\vec{R}_{\mathcal{S}}^{-1}, \tag{88}$$

where the right-hand side consists of the product of Mueller transforms by considering $\vec{R}_{\mathcal{S}}$ as a Mueller transform. Also note that the coordinate conversion matrix for Stokes vectors can be rewritten as:

$$\mathbf{C}_{\vec{\mathrm{F}}\to\vec{\mathrm{G}}} = \left[\overleftrightarrow{I}\right]^{\vec{\mathrm{F}}\to\vec{\mathrm{G}}}, \tag{89}$$

where $\overleftrightarrow{I}$ indicates the identity Mueller transform.

4 ANALYSIS ON STOKES VECTOR FIELDS

Sections 4.1 to 4.4 will provide descriptions for analysis on Stokes vector fields to help understand why naively applying conventional scalar SH to rendering with polarized lights fails. It will support the fact that scalar SH suffers from a singularity problem for Stokes vector fields, and the singularity problem violates rotation invariance.

In addition, Sections 4.3 and 4.4 provide some formal techniques that will be used for the proofs in Section 5.

4.1 Preliminaries: Continuity of Scalar and Tangent Vector Fields

For better intuition, we first introduce scalar and tangent vector fields, which are simpler types than Stokes vector fields. Observing the difference between Stokes vector fields and the simpler types of fields may help understand the challenges of Stokes vector fields.

Scalar fields. Continuity of a (scalar-valued) spherical function, or scalar field, $f : \hat{\mathbb{S}}^2 \to \mathbb{R}$ or $\mathbb{C}$ is well defined when considering $\hat{\mathbb{S}}^2$ as a smooth surface embedded in $\vec{\mathbb{R}}^3$. However, it is often more convenient to test the continuity of the spherical function written in spherical coordinates, $f(\theta, \phi)$. The $f : \hat{\mathbb{S}}^2 \to \mathbb{C}$ is continuous if and only if its spherical coordinates parameterization $f(\theta, \phi)$[11] is continuous on $[0, \pi] \times [0, 2\pi]$ and the following conditions hold.

$$f(0, \phi_1) = f(0, \phi_2), \quad f(\pi, \phi_1) = f(\pi, \phi_2), \quad f(\theta, 0) = f(\theta, 2\pi), \quad \forall \phi_1, \phi_2 \in [0, 2\pi] \text{ and } \forall \theta \in [0, \pi]. \quad (90)$$

Analogously, the continuity of spherical stokes-valued functions can be tested in the $[0, \pi] \times [0, 2\pi]$ parameterization domain in the later section, but it has different constraints from the above.

Tangent vector fields. Before dealing with Stokes-value spherical functions such as Stokes vectors as a function of propagation directions, we will first explain tangent vector fields on the sphere to show the analogy and difference between them.

A tangent vector field on the sphere $\vec{f} : \hat{\mathbb{S}}^2 \to \cup_{\hat{\omega} \in \hat{\mathbb{S}}^2} T_{\hat{\omega}} \hat{\mathbb{S}}^2$ is a function defined on the sphere $\hat{\mathbb{S}}^2$ of which each value at $\hat{\omega} \in \hat{\mathbb{S}}^2$ takes value from $\vec{f}(\hat{\omega}) \in T_{\hat{\omega}} \hat{\mathbb{S}}^2$, where $T_{\hat{\omega}} \hat{\mathbb{S}}^2$ denotes the tangent plane of $\hat{\mathbb{S}}^2$ at $\hat{\omega}$ defined by $T_{\hat{\omega}} \hat{\mathbb{S}}^2 := \left\{ v \in \vec{\mathbb{R}}^3 \mid \hat{\omega} \cdot v = 0 \right\}$.

As examples to help intuition of tangent vector fields, one can imagine a tangent vector field on the sphere as a wind velocity map on the earth or the gradient vector field of an omnidirectional image obtained by a fish-eye lens.

Representation under a coordinates system. Since a tangent vector field on the sphere takes a value from a different tangent plane at each point $\hat{\omega}$, representing the tangent vector field is more complicated than scalar fields. One common way is to use frame fields. A *frame field* on $\hat{\mathbb{S}}^2$, $\vec{\mathbf{F}}(\hat{\omega})$, is defined as a function maps (almost everywhere) each point $\hat{\omega} \in \hat{\mathbb{S}}^2$ to a frame $\vec{\mathbf{F}}(\hat{\omega}) \in \vec{\mathbb{F}}^3_{\hat{\omega}}$, which has $\hat{\omega}$ as the third axis, i.e., $\vec{\mathbf{F}}(\hat{\omega})[:, 3] = \hat{\omega}$. Note that frame fields are usually required to be continuous except at a zero-measure singularity (usually two points). Then a tangent vector field $\vec{f} : \mathbb{S}^2 \to \cup_{\hat{\omega} \in \mathbb{S}^2} T_{\hat{\omega}} \mathbb{S}^2$ can be represented as:

$$\vec{f}(\hat{\omega}) = a(\hat{\omega}) \vec{\mathbf{F}}(\hat{\omega})[:, 1] + b(\hat{\omega}) \vec{\mathbf{F}}(\hat{\omega})[:, 2],$$

for some scalar-valued spherical functions a and b. A usual way to select the $\theta\phi$ frame field is introduced in Equation (9), which is aligned to the spherical coordinates. Recall the formulae in more detail; it can be written

[11]For rigorous mathematics we need another symbol rather than f, which is defined on the sphere, but we use the symbol for better intuition.

as follows:

$$\vec{\mathbf{F}}_{\theta\phi}(\theta,\phi) \coloneqq \left[\hat{\theta},\hat{\phi},\hat{\omega}\right],$$
$$\text{where } \hat{\theta} \coloneqq \operatorname{normalize}\left(\frac{\partial\hat{\omega}_{\text{sph}}}{\partial\theta}\right) = \vec{\mathbf{F}}_g\left[\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right]^T,$$
$$\hat{\phi} \coloneqq \operatorname{normalize}\left(\frac{\partial\hat{\omega}_{\text{sph}}}{\partial\phi}\right) = \vec{\mathbf{F}}_g\left[-\sin\phi, \cos\phi, 0\right]^T. \tag{91}$$

Here $\vec{\mathbf{F}}_g$ indicates the global (world) frame, and $\hat{\omega}_{\text{sph}}$ indicates the spherical coordinate system specified by the global frame $\vec{\mathbf{F}}_g$ as defined by Equation (8).

Continuity of tangent vector fields. Unlike scalar fields, coordinate systems and frame fields raise discontinuity, which does not contain the original structure of the sphere $\mathbb{S}^2$, only testing the continuity of $a(\hat{\omega})$ and $b(\hat{\omega})$ is not enough to test the continuity of the vector field $\vec{f}$. The continuity of $\vec{f}$ is rewritten in terms of a and b as follows:

$$a \text{ and } b \text{ are continuous on } \hat{\mathbb{S}}^2 - S_{\vec{\mathbf{F}}},$$
$$\forall\hat{\omega}_s \in S_{\vec{\mathbf{F}}}, \quad \lim_{\hat{\omega}\to\hat{\omega}_s} a(\hat{\omega})\,\vec{\mathbf{F}}(\hat{\omega})\,[:,1] + b(\hat{\omega})\,\vec{\mathbf{F}}(\hat{\omega})\,[:,2] \text{ converges.}$$

where $S_{\vec{\mathbf{F}}} \subset \hat{\mathbb{S}}^2$ denotes the set of singularities of the frame $\vec{\mathbf{F}}$. Note that every frame field has singularities due to the Hairy ball theorem. For symbolic or numerical evaluation, the above must be reformulated into a coordinate system, usually a spherical one. We observe that the simplest case to describe this constraint occurs when singularities of the frame field are a subset of discontinuity of the coordinate system, for instance, $\theta = 0$ or π and $\phi = 0$ or 2π for the spherical coordinates. In this context, we investigate continuity conditions of several types of spherical functions in terms of spherical coordinates and $\theta\phi$ frame field.

While $\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})$ at $\hat{\omega} = \pm\hat{z}_g$ is considered to be *not defined*, it is more useful to consider that $\vec{\mathbf{F}}_{\theta\phi}(0 \text{ or } \pi, \phi)$ is defined depending on ϕ by directly substituting θ to Equation (91) as follows:

$$\vec{\mathbf{F}}_{\theta\phi}(0,\phi) = \vec{\mathbf{F}}_g \begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix} = \vec{\mathbf{F}}_g\mathbf{R}_{zy}(\phi,0), \quad \vec{\mathbf{F}}_{\theta\phi}(\pi,\phi) = \vec{\mathbf{F}}_g \begin{bmatrix}-\cos\phi & -\sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & -1\end{bmatrix} = \vec{\mathbf{F}}_g\mathbf{R}_{zy}(\phi,\pi). \tag{92}$$

Denoting $\mathbf{f} \coloneqq [a,b]^T$ under $\vec{\mathbf{F}}_{\theta\phi}$, the tangent vector field $\vec{f}$ is continuous if and only if $\mathbf{f}(\theta,\phi)$ is continuous on $[0,\pi]\times[0,2\pi]$ and:

$$\mathbf{f}(0,\phi_2) = \mathbf{R}_2(-\phi_2+\phi_1)\,\mathbf{f}(0,\phi_1), \quad \mathbf{f}(\pi,\phi_2) = \mathbf{R}_2(+\phi_2-\phi_1)\,\mathbf{f}(\pi,\phi_1), \quad \mathbf{f}(\theta,0) = \mathbf{f}(\theta,2\pi),$$
$$\text{for any } \phi_1,\phi_2 \in [0,2\pi] \text{ and } \theta \in [0,\pi], \tag{93}$$

which has different conditions from scalar fields.

4.2 Continuity of Stokes Vector Fields

Stokes vector fields on the sphere. Now, we can consider applying the advantages of spherical harmonics on spherical functions to polarized intensity. Then, we should first look into the spherical functions of Stokes vectors (or Stokes vector fields on the sphere). Different from the case of scalar radiance, but similar to tangent vector

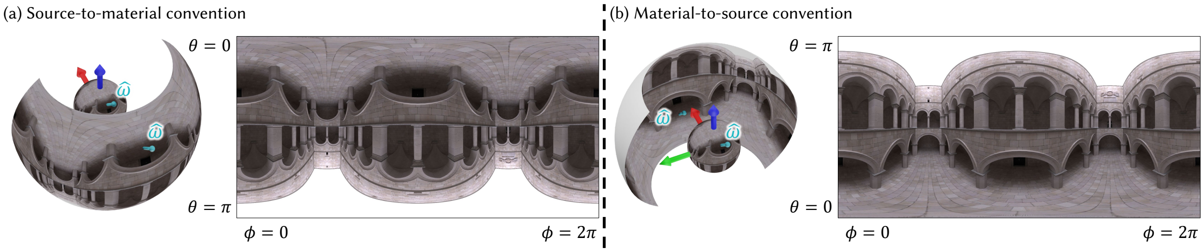

Fig. 8. We can unwrap an image on a sphere into an equirectangular image in two ways depending on a spherical point in the domain: (a) a ray propagation direction or (b) a light vector that points to a light source from a material (or observer). In our main paper, Main. Figure 6 follows the convention in (a) since it describes the general properties of spherical functions. The others Main Figures. 5, 4, 11, 16, and 18 follows the convention in (b) for better intuition since they describe environment map images. Note that θ and ϕ in our equations always indicate spherical coordinates of ray propagation directions so that the top row of the equirectangular image in (b) indicates $\theta = \pi$ while the one in (a) indicates $\theta = 0$.

fields, a Stokes vector field $\overset{\leftrightarrow}{f} \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}}\right)$ [12] has also the challenge that it evaluates given directions $\hat{\omega} \in \hat{\mathbb{S}}^2$ into values from different Stokes spaces $\overset{\leftrightarrow}{f}(\hat{\omega}) \in \mathcal{S}_{\hat{\omega}}$.

Representation under a coordinates system. Similar to tangent vector fields, we can represent a Stokes field into four components of scalar fields, but this cannot be done directly by applying local frames as linear operators on vectors. We must use the Stokes component conversion defined in Definition 3.2. Then, we can rewrite the Stokes vector field $\overset{\leftrightarrow}{f}$ as follows:

$$\left[\overset{\leftrightarrow}{f}(\hat{\omega})\right]^{\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})} = \begin{bmatrix} f_0(\theta,\phi) & f_1(\theta,\phi) & f_2(\theta,\phi) & f_3(\theta,\phi) \end{bmatrix}^T. \tag{94}$$

The continuity of $\overset{\leftrightarrow}{f}$ can be represented in terms of each component $f_0, \cdots, f_3$, and it yields different constraints at the singularities $\pm\hat{z}_g$ from both scalar and tangent vector fields. By denoting $\mathbf{f} \coloneqq [f_1, f_2]^T$,

$$\mathbf{f}(0,\phi_2) = \mathbf{R}_2(-2(\phi_2-\phi_1))\,\mathbf{f}(0,\phi_1), \quad \mathbf{f}(\pi,\phi_2) = \mathbf{R}_2(2(\phi_2-\phi_1))\,\mathbf{f}(\pi,\phi_1), \quad \mathbf{f}(\theta,0) = \mathbf{f}(\theta,2\pi), \\ \text{for any } \phi_1,\phi_2 \in [0,2\pi] \text{ and } \theta \in [0,\pi], \tag{95}$$

while f_0 and f_3 components are conventional scalar fields. Note that the first two constraints of f_1 and f_2 appear twice the components' rotation. From such different conditions, representing a Stokes vector field using a continuous scalar or tangent vector field yields a discontinuous Stokes vector field, which implies each type of field should have different types of continuous basis functions.

4.3 Stokes Vector Fields Operations

To discuss bases for Stokes vector fields, we should define several operations on Stokes vector fields. It can be done by generalizing scalar field operations in Section 2, based on Stokes vectors operations in Section 3. We define the inner product and rotations of Stokes vector fields as follows.

[12]Rigorously, it should be written as $\left\{\overset{\leftrightarrow}{f}\colon \hat{\mathbb{S}}^2 \to \cup_{\hat{\omega}\in\hat{\mathbb{S}}^2}\mathcal{S}_{\hat{\omega}} \mid \forall\hat{\omega}\in\hat{\mathbb{S}}^2, \overset{\leftrightarrow}{f}(\hat{\omega}) \in \mathcal{S}_{\hat{\omega}}\right\}$. But we write as the main text for the sake of simplicity and better intuition.

Definition 4.1: Inner product of Stokes vector fields

For Stokes vector fields $\overleftrightarrow{f}, \overleftrightarrow{g} : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}$, the inner product of them is defined as follows.

$$\left\langle \overleftrightarrow{f}, \overleftrightarrow{g} \right\rangle_{\mathcal{F}(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}})} := \int_{\hat{\mathbb{S}}^2} \left\langle \overleftrightarrow{f}(\hat{\omega}), \overleftrightarrow{g}(\hat{\omega}) \right\rangle_{\mathcal{S}} \mathrm{d}\hat{\omega}. \tag{96}$$

Definition 4.2: Rotation of Stokes vector fields

For $\vec{R} \in \overrightarrow{SO}(3)$, it can acts as $\vec{R}_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}}\right)\right)$, a linear operator on Stokes vector fields as follows.

$$\vec{R}_{\mathcal{F}}\left[\overleftrightarrow{f}\right](\hat{\omega}) = \vec{R}_{\mathcal{S}}\left[\overleftrightarrow{f}\left(\vec{R}^{-1}\hat{\omega}\right)\right], \quad \forall \overleftrightarrow{f} : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}. \tag{97}$$

Note that the inner product in Definition 4.1 is often written as simply $\left\langle \overleftrightarrow{f}, \overleftrightarrow{g} \right\rangle_{\mathcal{F}}$. The rotation defined in Definition 4.2 is illustrated in Figure 3(b).

4.4 Scalar SH to Stokes Vector Fields

Now, we will show two problems when using scalar SH to Stokes vectors: singularity and violation of rotation invariance.

4.4.1 Singularity. We first focus on the continuity condition for Stokes vector fields. Concretely, we can try to naïvely apply the scalar SH on each component $f_0 \dots f_3$ of the Stokes vector field with respect to the $\theta\phi$-frame field $\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})$ as

$$\overleftrightarrow{Y}^{(\text{naive})}_{lm0} := \begin{bmatrix} Y_{lm}(\hat{\omega}) \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})}, \dots, \overleftrightarrow{Y}^{(\text{naive})}_{lm3} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ Y_{lm}(\hat{\omega}) \end{bmatrix}_{\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})}, \tag{98}$$

which is considered as a basis, where $0 \le |m| \le l$. However, scalar SH satisfy

$$\begin{aligned} Y_{l0}(0, \phi) &= \text{const.} \neq 0, \\ Y_{l0}(\pi, \phi) &= \text{const.} \neq 0, \end{aligned} \tag{99}$$

and those conditions never satisfy the continuity condition of spin-2 functions in Equation (95). Thus, even if a continuous Stokes vector field $\overleftrightarrow{f}$ is given, its finite projection on the basis in Equation (98) up to $l \le l_{\max}$ is always discontinuous at $\pm\hat{z}_g$. This is a fundamentally different feature from how the scalar SH behaved on scalar fields, which always converts finite coefficients to continuous functions and has a smoothing role.

4.4.2 Rotation invariance violation. The singularity issue of basis functions is not only the presence of singularity itself but also the effects of the continuity of the basis function, which is a necessary condition for rotation invariance.

Note that $\overleftrightarrow{Y}^{(\text{naive})}_{l'01}(\hat{\omega})$ is discontinuous at $\hat{\omega} = \pm\hat{z}_g$. So when rotating it by $\vec{R} = \vec{R}_{y_g}\left(\frac{\pi}{2}\right)$, then the rotated basis $\vec{R}_{\mathcal{F}}\left[\overleftrightarrow{Y}^{(\text{naive})}_{l'01}\right]$ is discontinuous at $\hat{\omega} = \pm\hat{x}_g$. Thus, when decomposing it into a linear combination of the original basis $\overleftrightarrow{Y}^{(\text{naive})}_{l'mp}$, which is always continuous at $\hat{\omega} = \pm\hat{x}_g$, the linear combination must be an infinite sum to make such discontinuity since the finite sum of continuous functions is always continuous. Generally, it can be written as a coefficient matrix of the rotation as

$$\left\langle \overleftrightarrow{Y}^{(\text{naive})}_{lmp}, \vec{R}_{\mathcal{F}}\left[\overleftrightarrow{Y}^{(\text{naive})}_{l'm'p'}\right]\right\rangle_{\mathcal{F}} \neq 0, \text{ for } l \neq l', \tag{100}$$

where an inner product of two Stokes vector fields is defined in Definition 4.1

Recall that the rotation invariance of SH for scalar fields is represented as a block diagonal coefficient matrix in Equation (45). However, Equation (100) implies that the elements of the coefficient matrix at $l \neq l'$ are nonzero. This means SH for the Stokes vector field does not yield a block diagonal and violates the rotation invariance. For further validation related to rotation invariance, refer to Section 6.2 in the main paper.

4.5 Rotation Form of Stokes Vector Fields

Rather than unwrapping Stokes vector fields into spherical coordinates, the following formulation is sometimes useful in deriving our theory.

Definition 4.3: Rotation form of Stokes vector fields

Given a global frame $\vec{\mathbf{F}}_g$, for a spin-2 Stokes vector field $\overleftrightarrow{f} : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}$, its rotation form $f : \overrightarrow{SO}(3) \to \mathbb{C}$ is defined as follows.

$$f\left(\vec{R}\right) = \left[\overleftrightarrow{f}\left(\vec{R}\hat{z}_g\right)\right]^{\vec{R}\vec{\mathbf{F}}_g}. \tag{101}$$

Note that full Stokes vector fields can be similarly redefined as a function with codomain $\mathbb{R}^4$ rather than $\mathbb{C}$. Note that the following property is converse.

Proposition 4.4: Stokes vector fields from rotation forms

A function $f : \overrightarrow{SO}(3) \to \mathbb{C}$ can be a rotation form of a spin-2 Stokes vector field if and only if

$$f\left(\vec{R}\vec{R}\hat{z}_g(\psi)\right) = e^{-2i\psi} f\left(\vec{R}\right). \tag{102}$$

Note that it comes from the continuity condition of Stokes vector fields. For a function $f : \overrightarrow{SO}(3) \to \mathbb{R}^4$, the condition to be equivalent to a full Stokes vector can be obtained by substituting $e^{2i\psi}$ by the $\mathbf{C}_{\vec{\mathbf{F}}\to\vec{\mathbf{G}}}$ matrix.

4.6 Mueller Transform Fields

Similar to Stokes vector fields, we can also define a Mueller transform field as a function $\overleftrightarrow{M} : \hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2 \to \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$ which satisfies $\overleftrightarrow{M}(\hat{\omega}_i, \hat{\omega}_o) \in \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$.

We define the rotation of a Mueller transform field as follows.

Definition 4.5: Rotation of Mueller transform fields

For $\vec{R} \in \overrightarrow{SO}(3)$, it can acts as $\vec{R}_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2, \mathcal{M}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2, \mathcal{M}\right)\right)$, a linear operator on Mueller transform fields as follows.

$$\vec{R}_{\mathcal{F}}\left[\overleftrightarrow{M}\right](\hat{\omega}_i, \hat{\omega}_o) = \vec{R}_{\mathcal{M}}\left[\overleftrightarrow{M}\left(\vec{R}^{-1}\hat{\omega}_i, \vec{R}^{-1}\hat{\omega}_o\right)\right], \quad \forall \overleftrightarrow{f} : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}. \tag{103}$$

Note that it can be understood as a pBRDF obtained by rotating the material in a rendering context. Mueller transform fields are more discussed in later Section 5.5.

5 POLARIZED SPHERICAL HARMONICS FOR STOKES VECTOR FIELD

5.1 Spin-weighted Spherical Harmonics

Note that our definition of spin-weighted functions and SWSH may take a slightly different formulation than other literature, but still equivalent. We chose our formulation for convenience to derive our PSH theory.

Definition 5.1: Spin-weight s functions

Given a global frame $\vec{\mathbf{F}}_g$, $f : \overrightarrow{SO}(3)(3) \to \mathbb{C}$ (or $f : \vec{\mathbb{F}}^3 \to \mathbb{C}$) is called a *spin-weight s function* (or *spin-s function*, simply) if:

$$f\left(\vec{R}\vec{R}_{\hat{z}_g}(\psi)\right) = e^{-is\psi} f\left(\vec{R}\right) \text{ for any } \vec{R} \in \overrightarrow{SO}(3), \psi \in \mathbb{R}. \tag{104}$$

Equivalently, it can also be defined as $f : \hat{\mathbb{S}}^2 \to \bigcup_{\hat{\omega} \in \hat{\mathbb{S}}^2} \left(\mathbb{C} \times \vec{\mathbf{F}}_{\hat{\omega}}\right)/\sim$

$f(\hat{\omega}) \in \left(\mathbb{C} \times \vec{\mathbf{F}}_{\hat{\omega}}\right)/\sim$, where $\left(z_1, \vec{\mathbf{F}}_1\right) \sim \left(z_2, \vec{\mathbf{F}}_2\right)$ if and only if $\vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_1 \mathbf{R}_z(\psi)$ for some ϑ and $z_2 = e^{-is\psi} z_1$.

The condition also can be represented as:

$$f(\cos\psi\hat{x} - \sin\psi\hat{y}, \sin\psi\hat{x} + \cos\psi\hat{y}, \hat{z}) = e^{is\psi} f(\hat{x}, \hat{y}, \hat{z}), \tag{105}$$

by considering f as a function on $\vec{\mathbb{F}}^3$. Note that the definition of spin-s functions does not depend on the choice of global frame $\vec{\mathbf{F}}_g$. An important property is that there is a natural correspondence between spin-2 functions and Stokes-valued spherical functions by considering $f(\hat{x}, \hat{y}, \hat{z})$ as $s_1 + is_2$ where s_1 and s_2 are linear Stokes parameter for a ray along $\hat{z}$ with respect to the frame $[\hat{x}, \hat{y}, \hat{z}]$. One also observes that spin-0 and spin-1 functions are equivalent to the sphere's scalar and tangent vector fields, respectively.

Taking equivalent but slightly different orders to derive SWSH, we define SWSH as follows.

Definition 5.2: Spin-weighted spherical harmonics

The spin-weighted spherical harmonics with spin s, order l, and degree m is a spin-s function defined as follows:

$${}_sY_{lm}\left(\vec{R}\right) = (-1)^s \sqrt{\frac{2l+1}{4\pi}} D^{l*}_{m,-s}\left(\vec{R}\right). \tag{106}$$

Note that due to Proposition 2.6(6), SWSH becomes an orthonormal basis for spin-s functions, with a differential measure on $\hat{\mathbb{S}}^2$ following the definition through equivalence classes described in Definition 5.1.

Proposition 5.3: Spin-2 spherical harmonics in Stokes vector fields

Defining a spin-2 Stokes vector field $\overleftrightarrow{Y}_{lm}(\hat{\omega}) := \left[{}_2Y_{lm}\left(\vec{R}\right)\right]_{\vec{R}\vec{\mathbf{F}}_g}$, it becomes the well defined orthonormal basis for Stokes vectors fields, over scalar $\mathbb{C}$.

See also the rotation form of Stokes vectors fields discussed in Definition 4.3 and Proposition 4.4. Then a representation under the $\theta\phi$ frame field $\vec{\mathbf{F}}_{\theta\phi}$, which is introduced in the main paper, is defined as follows:

$${}_2Y_{lm}(\theta, \phi) := \left[\overleftrightarrow{Y}_{lm}(\theta, \phi)\right]^{\vec{\mathbf{F}}_{\theta\phi}(\theta,\phi)}. \tag{107}$$

Note that our main paper introduces the function in Equation (107) first and then derives the formulation in Proposition 5.3 later to start from numerically measurable quantity, which is regarded more practical.

5.2 Converting Between $\mathbb{R}^2$ and $\mathbb{C}$

we defines symbols to convert $\mathbb{C}$ and $\mathbb{R}^2$ or $\mathbb{R}^{2\times 2}$.

$$\mathbb{C}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \coloneqq x + yi \in \mathbb{C}, \tag{108}$$

$$\mathbb{R}^2\left(x + yi\right) \coloneqq \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2, \tag{109}$$

$$\mathbb{R}^{2\times 2}\left(x + yi\right) \coloneqq \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \in \mathbb{R}^{2\times 2}. \tag{110}$$

Then we get:

$$\mathbb{R}^2\left(z_1\right)^T \mathbb{R}^{2\times 2}\left(z_2\right) \cdots \mathbb{R}^{2\times 2}\left(z_{n-1}\right) \mathbb{R}^2\left(z_n\right) = \Re \left\langle z_1, z_2 \cdots z_n \right\rangle_{\mathbb{C}} = \Re\left(z_1^* z_2 \cdots z_n\right). \tag{111}$$

Complex pair separation. We observe that Equations (108) and (109) are the inverses of each other, but the function in Equation (110) has not the inverse since it is not surjective. However, we found that any 2×2 real matrix $\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ can be represented by two complex numbers as follows:

$$\begin{aligned} \mathbf{M} &= \mathbb{R}^{2\times 2}\left(\mathbb{C}_{\text{iso}}\left(\mathbf{M}\right)\right) + \mathbb{R}^{2\times 2}\left(\mathbb{C}_{\text{conj}}\left(\mathbf{M}\right)\right)\mathbf{J}, \\ \text{where } \mathbb{C}_{\text{iso}}\left(\mathbf{M}\right) &\coloneqq \frac{m_{11} + m_{22}}{2} + \frac{m_{21} - m_{12}}{2} i, \\ \mathbb{C}_{\text{conj}}\left(\mathbf{M}\right) &\coloneqq \frac{m_{11} - m_{22}}{2} + \frac{m_{21} + m_{12}}{2} i, \\ \mathbf{J} &\coloneqq \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{aligned} \tag{112}$$

We call $\mathbb{C}_{\text{iso}}\left(\mathbf{M}\right)$ and $\mathbb{C}_{\text{conj}}\left(\mathbf{M}\right)$ the *isomorphic part* and the *conjugation part of* $\mathbf{M}$, respectively.

The matrix $\mathbf{J}$ acts on all right complex representations as complex conjugation, i.e.,

$$\mathbf{J}\mathbb{R}^2\left(z\right) = \mathbb{R}^2\left(z^*\right). \tag{113}$$

In general,

$$\begin{aligned} &\mathbb{R}^2\left(z_1\right)^T \mathbb{R}^{2\times 2}\left(z_2\right) \cdots \mathbb{R}^{2\times 2}\left(z_d\right) \mathbf{J}\mathbb{R}^{2\times 2}\left(z_{d+1}\right) \cdots \mathbb{R}^{2\times 2}\left(z_{n-1}\right) \mathbb{R}^2\left(z_n\right) \\ &= \mathbb{R}^2\left(z_1\right)^T \mathbb{R}^{2\times 2}\left(z_2 \cdots z_d\right) \mathbf{J}\mathbb{R}^2\left(z_{d+1} \cdots z_n\right) = \Re \left\langle z_1, z_2 \cdots z_d z_{d+1}^* \cdots z_n^* \right\rangle_{\mathbb{C}} \end{aligned} \tag{114}$$

Please be careful that this fact cannot be reduced to a product of $\mathbf{J}$ and a single 2×2 matrix, i.e.,

$$\mathbf{J}\mathbb{R}^{2\times 2}\left(z\right) \neq \mathbb{R}^{2\times 2}\left(z^*\right), \tag{115}$$

since $\mathbf{J}$ cannot be $\mathbb{R}^{2\times 2}\left(z\right)$ for some $z \in \mathbb{C}$. Thus, we observe that Equation (114) should be obtained by contracting $\mathbb{R}^{2\times 2}\left(z_{d+1}\right) \cdots \mathbb{R}^{2\times 2}\left(z_{n-1}\right) \mathbb{R}^2\left(z_n\right) = \mathbb{R}^2\left(z_{d+1} \cdots z_n\right)$ first, and followed by applying Equation (113)

Complex indexing formulae. Due to the complexity of our derivation, such as viewing a function space as a linear space over scalar both $\mathbb{R}$ or $\mathbb{C}$, The following conversion equations will be useful. We call them complex indexing formulae.

$$\mathbf{Mat}\left[\Re\left(i^{1-p}z\right) \mid p=1,2\right]=\begin{bmatrix}\Re z\\ \Im z\end{bmatrix}=\mathbb{R}^{2}\left(z\right), \tag{116}$$

$$\mathbf{Mat}\left[\Re\left(i^{p-1}z\right) \mid p=1,2\right]=\begin{bmatrix}\Re z^{*}\\ \Im z^{*}\end{bmatrix}=\mathbb{R}^{2}\left(z^{*}\right), \tag{117}$$

$$\mathbf{Mat}\left[\Re\left(i^{p_i-p_o}z\right) \mid p_o,p_i=1,2\right]=\begin{bmatrix}\Re z & -\Im z\\ \Im z & \Re z\end{bmatrix}=\mathbb{R}^{2\times 2}\left(z\right), \tag{118}$$

$$\mathbf{Mat}\left[\Re\left(i^{2-p_i-p_o}z\right) \mid p_o,p_i=1,2\right]=\begin{bmatrix}\Re z & \Im z\\ \Im z & -\Re z\end{bmatrix}=\mathbb{R}^{2\times 2}\left(z\right)\mathbf{J}. \tag{119}$$

5.3 Polarized Spherical Harmonics

5.3.1 Discussion on real coefficient formulation. Note that we already discussed the necessity of our real coefficient formulation for spin-2 components for our PSH in the main paper in terms of complex pair separation, which is described both in the main paper and this document through Equations (112) to (115). Now, we discuss our real coefficient formulation for spin-0 components. We now have two choices when we fix spin-2 coefficients as $\mathbb{R}^2$. Using a basis $\delta_{p0}Y^R_{lm} \oplus \left(\delta_{p1}\overleftrightarrow{Y}_{lm1} + \delta_{p2}\overleftrightarrow{Y}_{lm2}\right) \oplus \delta_{p3}Y^R_{lm}$ and coefficients in $\mathbb{R}^4$, or using a basis $\delta_{p0}Y^C_{lm} \oplus \left(\delta_{p1}\overleftrightarrow{Y}_{lm1} + \delta_{p2}\overleftrightarrow{Y}_{lm2}\right) \oplus \delta_{p3}Y^C_{lm}$ and coefficients in $\mathbb{C} \oplus \mathbb{R}^2 \oplus \mathbb{C}$. While the former one, which will be selected our polarized spherical harmonics basis in Proposition 5.4, clearly implies that it encodes general $\mathbb{R}$-linear operators on $\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}}\right)$ into 4×4 real matrices of coefficients for fixed l and m indices, the later one cannot well define coefficient matrices. First, $\mathbb{R}$-linear operators on $\mathbb{C} \oplus \mathbb{R}^2 \oplus \mathbb{C}$ belong to $\left(\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2\right)^2$, which requires 2×2 real coefficients for fixed l and m indices to represent operators from s_0 components to s_0 components. It contains too much redundant information to describe real-valued data from the original angular domain. It is not even compatible with conventional formulation where SH encodes linear operators on scalar fields to a coefficient simply in $\mathbb{R}$ or $\mathbb{C}$ for fixed l and m indices. As another choice, if one tries to define a coefficient matrix with mixed entry types, $\mathbb{C}$ and $\mathbb{R}$, we cannot define it as closed under matrix multiplication. If taking a product of two such matrices, complex values in spin 0-to-0 submatrices make spin 0-to-2 and 2-to-0 submatrices become complex. Finally, they make spin 2-to-2 submatrices become complex when multiplying another matrix again. It yields a contradiction.

5.3.2 Polarized spherical harmonics. As discussed in the previous section, we define our polarized SH by combining spin-0 and spin-2 SH with real coefficient formulation.

Proposition 5.4: Polarized spherical harmonics

With an index set

$$I_{\mathrm{PSH}} = \left\{(l, m, p) \in \mathbb{Z}^2 \mid |m| \le l, 0 \le p < 4, \text{ and } [\text{if } p = 1, 2 \text{ then } l \ge 2]\right\}, \tag{120}$$

$\overleftrightarrow{Y}_{lmp}$'s are an orthonormal basis for the linear space of Stokes vector fields $\left\{f : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}\right\}$ over the scalar $\mathbb{R}$, where

$$\overleftrightarrow{Y}_{lm0}(\hat{\omega}) = \begin{bmatrix} Y^R_{lm}(\hat{\omega}) \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})}, \overleftrightarrow{Y}_{lm1}(\hat{\omega}) = \begin{bmatrix} 0 \\ \Re\left[{}_2Y_{lm}(\hat{\omega})\right] \\ \Im\left[{}_2Y_{lm}(\hat{\omega})\right] \\ 0 \end{bmatrix}_{\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})}, \overleftrightarrow{Y}_{lm2}(\hat{\omega}) = \begin{bmatrix} 0 \\ -\Im\left[{}_2Y_{lm}(\hat{\omega})\right] \\ \Re\left[{}_2Y_{lm}(\hat{\omega})\right] \\ 0 \end{bmatrix}_{\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})}, \overleftrightarrow{Y}_{lm3}(\hat{\omega}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Y^R_{lm}(\hat{\omega}) \end{bmatrix}_{\vec{\mathbf{F}}_{\theta\phi}(\hat{\omega})}. \tag{121}$$

Note that it can be rewritten as $\overleftrightarrow{Y}_{lmp} = \delta_{p0} Y^R_{lm}(\hat{\omega}) \oplus \left(\delta_{p1}\overleftrightarrow{Y}_{lm1} + \delta_{p2}\overleftrightarrow{Y}_{lm2}\right) \oplus \delta_{p3} Y^R_{lm}$. Also note that taking spin-2 Stokes vector from $\overleftrightarrow{Y}_{lm1}$ and $\overleftrightarrow{Y}_{lm2}$, $\overleftrightarrow{Y}_{lm} = \overleftrightarrow{Y}_{lm1}$ and $i\overleftrightarrow{Y}_{lm} = \overleftrightarrow{Y}_{lm2}$. The following formula is useful to derive our linear operator formulations through a few equations rather than enumerating each indices p_i and p_o. Using Equation (116), for $p = 1, 2$,

$$\overleftrightarrow{Y}_{lmp} = 0 \oplus \left[i^{p-1}\,{}_2Y_{lm}\right]_{\vec{\mathbf{F}}_{\theta\phi}} \oplus 0. \tag{122}$$

5.4 Rotation of Polarized Spherical Harmonics

Here, we provide the statement describing the PSH coefficient matrices of rotations on Stokes vector fields and its proof.

Proposition 5.5: Rotation coefficients of PSH

The coefficient matrices of a rotation transform $\vec{R} \in \overrightarrow{SO}(3)$ acting on the function space of Stokes vector fields, $\vec{R}_{\mathcal{F}}$, is evaluated as follows.

$$\begin{aligned} &\mathbf{Mat}\left[\left\langle \overleftrightarrow{Y}_{l_o m_o p_o}, \vec{R}_{\mathcal{F}}\left[\overleftrightarrow{Y}_{l_i m_i p_i}\right]\right\rangle_{\mathcal{F}} \mid p_0, p_i = 0, \cdots, 3\right] \\ &= \delta_{l_i l_o} \begin{bmatrix} D^{l,R}_{m_o m_i}\left(\vec{R}\right) & 0 & 0 & 0 \\ 0 & \Re D^{l,C}_{m_o m_i}\left(\vec{R}\right) & -\Im D^{l,C}_{m_o m_i}\left(\vec{R}\right) & 0 \\ 0 & \Im D^{l,C}_{m_o m_i}\left(\vec{R}\right) & \Re D^{l,C}_{m_o m_i}\left(\vec{R}\right) & 0 \\ 0 & 0 & 0 & D^{l,R}_{m_o m_i}\left(\vec{R}\right) \end{bmatrix} \\ &= \delta_{l_i l_o} \begin{bmatrix} D^{l,R}_{m_o m_i}\left(\vec{R}\right) & \mathbf{0}_{1\times 2} & 0 \\ \mathbf{0}_{2\times 1} & \mathbb{R}^{2\times 2}\left(D^{l,C}_{m_o m_i}\left(\vec{R}\right)\right) & \mathbf{0}_{2\times 1} \\ 0 & \mathbf{0}_{1\times 2} & D^{l,R}_{m_o m_i}\left(\vec{R}\right) \end{bmatrix}. \end{aligned} \tag{123}$$

Proof: Relation between spin-weighted spherical harmonics and Wigner-D matrices:

$$D^l_{m,-s}(\phi,\theta,\psi) = (-1)^s \sqrt{\frac{4\pi}{2l+1}}\, {}_sY^*_{lm}(\theta,\phi)\, e^{si\psi}. \tag{124}$$

Using rotation matrices and $\hat{z}$:

$$D^l_{m,-s}\left(\vec{R}\right) = (-1)^s \sqrt{\frac{4\pi}{2l+1}}\, {}_sY^*_{lm}\left(\vec{R}\hat{z}\right) e^{si\gamma_{zyz}\left(\vec{R}\right)}, \tag{125}$$

where $\gamma_{zyz}\left(\vec{R}\right)$ indicates an angle γ such that $\vec{R} = \vec{R}_{\hat{z}_g\hat{y}_g\hat{z}_g}(\alpha,\beta,\gamma)$.
Rotated basis can be evaluated as:

$$\left[\left(\vec{R}_{\mathcal{F}}\overset{\leftrightarrow}{Y}_{lmp}\right)\left(\vec{R}'\hat{z}\right)\right]^{\vec{R}'\vec{\mathrm{F}}} = \left[\overset{\leftrightarrow}{Y}_{lmp}\left(\vec{R}^{-1}\vec{R}'\hat{z}\right)\right]^{\vec{R}^{-1}\vec{R}'\vec{\mathrm{F}}} = \sqrt{\frac{2l+1}{4\pi}}\mathbb{R}^2\left[i^{p-1}D^{l*}_{m,-2}\left(\vec{R}^{-1}\vec{R}'\right)\right] \tag{126}$$

$$\begin{aligned}\left\langle \overset{\leftrightarrow}{Y}_{l'm'p'}, \vec{R}_{\mathcal{F}}\left[\overset{\leftrightarrow}{Y}_{lmp}\right]\right\rangle &= \frac{\sqrt{(2l+1)(2l'+1)}}{8\pi^2}\mathfrak{R}\int_{SO(3)} i^{p-p'} D^{l'}_{m',-2}(S)\, D^{l*}_{m,-2}\left(R^{-1}S\right) \mathrm{d}S \\ &= \delta_{ll'}\delta_{mm'}\mathfrak{R}\left[i^{p-p'}D^{l*}_{mm'}\left(R^{-1}\right)\right] = \delta_{ll'}\delta_{mm'}\mathfrak{R}\left[i^{p-p'}D^{l}_{m'm}(R)\right] \\ &= \delta_{ll'}\delta_{mm'}\left(\mathbb{R}^{2\times 2}\circ D^l_{m'm}(R)\right)_{p'p}.\end{aligned} \tag{127}$$

See also Boyle [2013]. □

5.5 Linear Operators (pBRDF, Radiance Transfer)

A linear operator on Stokes fields is characterized as a function of two directions into Mueller spaces.

Definition 5.6: Linear operators and kernels

Suppose there is a Mueller transform field $\overset{\leftrightarrow}{K}\colon \mathbb{S}^2\times\mathbb{S}^2 \to \mathcal{M}_{\hat{\omega}_i\to\hat{\omega}_o}$. The *linear operator of the kernel* $\overset{\leftrightarrow}{K}$, denoted by $\overset{\leftrightarrow}{K}_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}_i}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}_o}\right)\right)$, is defined as follows:

$$\forall \vec{s} \in \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}_{\hat{\omega}}\right), \quad \overset{\leftrightarrow}{K}_{\mathcal{F}}\left[\vec{s}\right](\hat{\omega}_i) = \int_{\hat{\mathbb{S}}^2} \overset{\leftrightarrow}{K}(\hat{\omega}_i,\hat{\omega}_o)\,\vec{s}(\hat{\omega}_i)\,\mathrm{d}\hat{\omega}_i. \tag{128}$$

If a linear operator $\overset{\leftrightarrow}{K}_{\mathcal{F}}$ is given first, a Mueller field $\overset{\leftrightarrow}{K}$ satisfying the above equation is called the *kernel* of the operator $\overset{\leftrightarrow}{K}_{\mathcal{F}}$.

A linear operator on Stokes fields can also be written as a function of two rotation transforms, similar to rotation forms for Stokes vector fields.

Definition 5.7: Rotation form of a Mueller transform field

The *rotation form of the Mueller transform field* $\overleftrightarrow{K}\colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$ (or the *rotation form of the operator* $\overleftrightarrow{K}_{\mathcal{F}}$) is defined as:

$$\begin{aligned} \tilde{\mathbf{K}}\colon \overrightarrow{SO}(3) \times \overrightarrow{SO}(3) &\to \mathbb{R}^{4\times 4}, \\ \tilde{K}\left(\vec{R}_i, \vec{R}_o\right) &= \left[\overleftrightarrow{K}\left(\vec{R}_i \hat{z}, \vec{R}_o \hat{z}\right)\right]^{\vec{R}_i \vec{\mathbf{F}}_e \to \vec{R}_o \vec{\mathbf{F}}_e}. \end{aligned} \tag{129}$$

Conversely, when a function $\tilde{\mathbf{K}}\colon \overrightarrow{SO}(3) \times \overrightarrow{SO}(3) \to \mathbb{R}^{4\times 4}$ is given, it can be the rotation form of a Mueller transform field if and only if it satisfies the following constraints:

$$\tilde{K}\left(\vec{R}_i \vec{R}_z(\psi_1), \vec{R}_o \vec{R}_z(\psi_2)\right) = \mathbf{R}_{1:2}(-2\psi_2)\, \tilde{K}\left(\vec{R}_i, \vec{R}_o\right) \mathbf{R}_{1:2}(2\psi_1). \tag{130}$$

We found that applying a linear operator to a Stokes vector field can be done in rotation forms of the Mueller transform field and the Stokes vector field.

Proposition 5.8: Applying linear operator in rotation forms

$\overleftrightarrow{f} : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}$ and $\overleftrightarrow{K} : \hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2 \to \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$ are a Stokes vector field and Mueller transform field, respectively. The rotation forms of $\overleftrightarrow{f}$ and $\overleftrightarrow{K}$ are denoted by and $\mathbf{M}$, respectively. Then the rotation form of $\overleftrightarrow{K}_{\mathcal{F}}\left[\overleftrightarrow{f}\right]$ can be evaluated as follows.

$$\mathbf{g}\left(\vec{R}_o\right) = \frac{1}{2\pi} \int_{\overrightarrow{SO}(3)} \mathbf{K}\left(\vec{R}_i, \vec{R}_o\right) \mathbf{f}\left(\vec{R}_i\right) \mathrm{d}\vec{R}_i,$$

where $\mathbf{g}$ denotes the rotation form of the resulting Stokes vector field.

Proof: By definition $\mathbf{g}$ is obtained as:

$$\mathbf{g}\left(\vec{R}_o\right) = \left[\overleftrightarrow{K}_{\mathcal{F}}\left[\overleftrightarrow{f}\right]\left(\vec{R}_o \hat{z}_g\right)\right]^{\vec{R}_o \vec{\mathbf{F}}_g}.$$

The term inside $[\cdot]$ can be obtained as follows using the integral conversion in Equation (15):

$$\int_{\hat{\mathbb{S}}^2} \overleftrightarrow{K}\left(\hat{\omega}_i, \vec{R}_o \hat{z}_g\right) \overleftrightarrow{f}(\hat{\omega}_i)\, \mathrm{d}\hat{\omega}_i = \frac{1}{2\pi} \int_{\overrightarrow{SO}(3)} \overleftrightarrow{K}\left(\vec{R}_i \hat{z}_g, \vec{R}_o \hat{z}_g\right) \overleftrightarrow{f}\left(\vec{R}_i \hat{z}_g\right) \mathrm{d}\vec{R}_i.$$

Note that the integrand on the right hand side is $\left[\mathbf{K}\left(\vec{R}_i, \vec{R}_o\right) \mathbf{f}\left(\vec{R}_i\right)\right]_{\vec{R}_o \vec{\mathbf{F}}_g}$. Substituting all equations into the first one,

$$\mathbf{g}\left(\vec{R}_o\right) = \frac{1}{2\pi} \int_{\overrightarrow{SO}(3)} \mathbf{K}\left(\vec{R}_i, \vec{R}_o\right) \mathbf{f}\left(\vec{R}_i\right) \mathrm{d}\vec{R}_i, \tag{131}$$

which also yields

$$\overleftrightarrow{g}\left(\vec{R}_o \hat{z}_g\right) = \frac{1}{2\pi} \left[\int_{\overrightarrow{SO}(3)} \mathbf{K}\left(\vec{R}_i, \vec{R}_o\right) \mathbf{f}\left(\vec{R}_i\right) \mathrm{d}\vec{R}_i\right]_{\vec{R}_o \vec{\mathbf{F}}_g}. \tag{132}$$

□

Definition 5.9: Complex form of a Mueller transform field

The *complex form of* a Mueller transform field $\overleftrightarrow{K} \colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$ is defined as ten functions $\tilde{K}_{0|3,0|3} \colon \hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2 \to \mathbb{R}$ and $\tilde{K}_{0|3,\mathrm{p}}, \tilde{K}_{\mathrm{p},0|3}, \tilde{K}_{\mathrm{ppi}}, \tilde{K}_{\mathrm{ppc}} \colon \hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2 \to \mathbb{C}$ which satisfy:

$$\left[\overleftrightarrow{K}\left(\hat{\omega}_i, \hat{\omega}_o\right)\right]^{\vec{\mathrm{F}}_{\theta\phi}(\hat{\omega}_i) \to \vec{\mathrm{F}}_{\theta\phi}(\hat{\omega}_o)} = \begin{bmatrix} \tilde{K}_{00} & \mathbb{R}^2\left(\tilde{K}_{0\mathrm{p}}\right)^T & \tilde{K}_{03} \\ \mathbb{R}^2\left(\tilde{K}_{\mathrm{p}0}\right) & \mathbb{R}^{2\times 2}\left(\tilde{K}_{\mathrm{ppi}}\right) + \mathbb{R}^{2\times 2}\left(\tilde{K}_{\mathrm{ppc}}\right)\mathbf{J} & \mathbb{R}^2\left(\tilde{K}_{\mathrm{p}3}\right) \\ \tilde{K}_{30} & \mathbb{R}^2\left(\tilde{K}_{3\mathrm{p}}\right)^T & \tilde{K}_{33} \end{bmatrix}. \tag{133}$$

Note that we omit the variables $(\hat{\omega}_i, \hat{\omega}_o)$ for each $\tilde{K}$ component for simplicity.

In Definition 5.9, each component should satisfy the following quantities and functions on the rotation group, which satisfy the following can be complex forms of a Mueller transform field, conversely.

$$\tilde{K}_{0|3,p}\left(\vec{R}_i \vec{R}_z(\psi), \vec{R}_o\right) = \tilde{K}_{0|3,p}\left(\vec{R}_i, \vec{R}_o\right) e^{-2\psi i} \tag{134}$$

$$\tilde{K}_{p,0|3}\left(\vec{R}_i, \vec{R}_o \vec{R}_z(\psi)\right) = \tilde{K}_{p,0|3}\left(\vec{R}_i, \vec{R}_o\right) e^{-2\psi i} \tag{135}$$

$$\tilde{K}_{ppa}\left(\vec{R}_i \vec{R}_z(\psi_1), \vec{R}_o \vec{R}_z(\psi_2)\right) = \tilde{K}_{ppa}\left(\vec{R}_i, \vec{R}_o\right) e^{-2(\psi_2 - \psi_1) i} \tag{136}$$

$$\tilde{K}_{ppb}\left(\vec{R}_i \vec{R}_z(\psi_1), \vec{R}_o \vec{R}_z(\psi_2)\right) = \tilde{K}_{ppa}\left(\vec{R}_i, \vec{R}_o\right) e^{-2(\psi_2 + \psi_1) i} \tag{137}$$

The coefficient matrix of a linear operator on Stokes vector fields can be defined and evaluated by 16 integral formulae obtained by directly extending Proposition 1.4. However, we found that they can be evaluated with fewer formulae using the complex form of the Mueller transform field.

Proposition 5.10: Coefficient matrix using the complex form of a Mueller field

The polarized spherical harmonics coefficients $\mathrm{M}_{l_o m_o p_p, l_i m_i p_i} := \left\langle \overleftrightarrow{Y}_{l_o m_o p_p}, \overleftrightarrow{M}_{\mathcal{F}}\left[\overleftrightarrow{Y}_{l_i m_i p_i}\right]\right\rangle$ of a linear operator with the kernel $\overleftrightarrow{M} \colon \mathbb{S}^2 \times \mathbb{S}^2 \to \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$ is evaluated using the complex form of $\overleftrightarrow{M}$ as follows:

$$\begin{aligned}
\mathrm{M}_{l_o m_o 0|3, l_i m_i 0|3} &= \int_{\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2} Y^R_{l_o m_o}\left(\hat{\omega}_o\right) \tilde{M}_{0|3,0|3}\left(\hat{\omega}_i, \hat{\omega}_o\right) Y^R_{l_i m_i}\left(\hat{\omega}_i\right) \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o, \\
\begin{bmatrix} \mathrm{M}_{l_o m_o 0|3, l_i m_i 1} & \mathrm{M}_{l_o m_o 0|3, l_i m_i 2} \end{bmatrix} &= \mathbb{R}^2\left(\int_{\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2} Y^R_{l_o m_o}\left(\hat{\omega}_o\right) \tilde{M}_{0|3,\mathrm{p}}\left(\hat{\omega}_i, \hat{\omega}_o\right) {}_2Y^*_{l_i m_i}\left(\hat{\omega}_i\right) \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o\right)^T, \\
\begin{bmatrix} \mathrm{M}_{l_o m_o 1, l_i m_i 0|3} \\ \mathrm{M}_{l_o m_o 2, l_i m_i 0|3} \end{bmatrix} &= \mathbb{R}^2\left(\int_{\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2} {}_2Y^*_{l_o m_o}\left(\hat{\omega}_o\right) \tilde{M}_{\mathrm{p},0|3}\left(\hat{\omega}_i, \hat{\omega}_o\right) Y^R_{l_i m_i}\left(\hat{\omega}_i\right) \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o\right), \\
\begin{bmatrix} \mathrm{M}_{l_o m_o 1, l_i m_i 1} & \mathrm{M}_{l_o m_o 1, l_i m_i 2} \\ \mathrm{M}_{l_o m_o 2, l_i m_i 1} & \mathrm{M}_{l_o m_o 2, l_i m_i 2} \end{bmatrix} &= \mathbb{R}^{2\times 2}\left(\int_{\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2} {}_2Y^*_{l_o m_o}\left(\hat{\omega}_o\right) \tilde{M}_{\mathrm{ppi}}\left(\hat{\omega}_i, \hat{\omega}_o\right) {}_2Y_{l_i m_i}\left(\hat{\omega}_i\right) \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o\right) \\
&\quad + \mathbb{R}^{2\times 2}\left(\int_{\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2} {}_2Y^*_{l_o m_o}\left(\hat{\omega}_o\right) \tilde{M}_{\mathrm{ppc}}\left(\hat{\omega}_i, \hat{\omega}_o\right) {}_2Y^*_{l_i m_i}\left(\hat{\omega}_i\right) \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o\right)\mathbf{J}.
\end{aligned} \tag{138}$$

Proof: Proof can be done by using the definition of the complex forms. Here, we clarify derivation steps for spin 2-to-2 components, $p_i, p_o = 1, 2$, which additionally utilize our complex pair separation and complex indexing formulae in Equations (118) and (119) as

$$\begin{aligned}\mathrm{f}\left[l_i m_i p_i, l_o m_o p_o\right] &= \left\langle \overleftrightarrow{\mathrm{Y}}_{l_o m_o p_o}, \overleftrightarrow{M}\overleftrightarrow{\mathrm{Y}}_{l_i m_i p_i} \right\rangle \\ &= \Re \int_{S^2\times S^2} i^{1-p_o}\, {}_2Y^*_{l_o m_o}\left(\hat{\omega}_o\right)\left[i^{p_i-1}\tilde{M}_{\mathrm{ppi}}\, {}_2Y_{l_i m_i} + i^{1-p_i}\tilde{M}_{\mathrm{ppi}}\, {}_2Y^*_{l_i m_i}\right]\mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o \\ &= \mathbb{R}^{2\times 2}\left[\int_{S^2\times S^2} \tilde{M}_{\mathrm{ppi}}\left(\hat{\omega}_i, \hat{\omega}_o\right) {}_2Y^*_{l_o m_o}\left(\hat{\omega}_o\right) {}_2Y_{l_i m_i}\left(\hat{\omega}_i\right)\mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o\right] \\ &+ \mathbb{R}^{2\times 2}\left[\int_{S^2\times S^2} \tilde{M}_{\mathrm{ppc}}\left(\hat{\omega}_i, \hat{\omega}_o\right) {}_2Y^*_{l_o m_o}\left(\hat{\omega}_o\right) {}_2Y^*_{l_i m_i}\left(\hat{\omega}_i\right)\mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o\right]\mathbf{J}.\end{aligned} \tag{139}$$

□

5.6 Reflection Operator

To adapt Section 2.7 to our PSH, we first define a reflection operator $T_{\mathcal{S}} \in \mathcal{L}\left(\mathcal{S}, \mathcal{S}\right)$ with respect to $\hat{z}_g$ as follows.

$$\left[T_{\mathcal{S}}\left(\overleftrightarrow{s}\right)\right]^{\vec{R}_{zyz}(\alpha,\beta,\gamma)\vec{\mathrm{F}}_g} = \left(\left[\overleftrightarrow{s}\right]^{\vec{R}_{zyz}(\alpha,\pi-\beta,0|\pi-\gamma)\vec{\mathrm{F}}_g}\right)^*. \tag{140}$$

Note that it can be understood by flipping the double-sided arrow, which visualizes a Stokes vector. It is also equivalent to perfect mirror reflection by the dielectric material of infinite index of refraction.

It can also act on Stokes vector fields, and its PSH coefficients are obtained in the following steps. Using ZYZ Euler angles for rotations,

$$\left\langle \overleftrightarrow{\mathrm{Y}}_{lm}, T_{\mathcal{F}}\left[\overleftrightarrow{\mathrm{Y}}_{l'm'}\right]\right\rangle = \int_{\mathbb{S}^2} {}_2Y^*_{lm}\left(\alpha,\beta,\gamma\right) {}_2Y^*_{l'm'}\left(\alpha, \pi-\beta, 0|\pi-\gamma\right)\mathrm{d}\hat{\omega}. \tag{141}$$

Note that it is constant for γ. For Wigner-D form:

$$\sqrt{\frac{(2l+1)\left(2l'+1\right)}{16\pi^2}}\frac{1}{2\pi}\int_{SO(3)} D^l_{m,-2}\left(\alpha,\beta,\gamma\right) D^{l'}_{m',-2}\left(\alpha,\pi-\beta,0|\pi-\gamma\right)\mathrm{d}\vec{R}. \tag{142}$$

By symmetry of small-D $d^l_{mm'}$ and Wigner-D,

$$D^{l'}_{m',-2}\left(\alpha,\pi-\beta,0|\pi-\gamma\right) = (-1)^{l'+m'} D^{l'}_{m',2}\left(\alpha,\beta,\gamma\right) = (-1)^{l'} D^{l'*}_{-m',-2}\left(\alpha,\beta,\gamma\right). \tag{143}$$

Substituting the above, the orthogonality of Wigner D-functions yields:

$$\left\langle \overleftrightarrow{\mathrm{Y}}_{lm}, T_{\mathcal{F}}\left[\overleftrightarrow{\mathrm{Y}}_{l'm'}\right]\right\rangle = (-1)^l \delta_{ll'}\delta_{m,-m'}. \tag{144}$$

5.7 Triple Product of SWSH

There are special symbols to represent the SWSH triple product. Spin-0 SH can be written as:

$$\int_{\hat{\mathbb{S}}^2} Y^*_{l_1 m_1} Y_{l_2 m_2} Y_{l_3 m_3}\mathrm{d}\hat{\omega} = (-1)^{m_1}\sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix}, \tag{145}$$

where the symbol $\begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3 \end{pmatrix}$ is called a *Wigner 3-j symbol*. Then the triple product of two spin-2 SH and one spin-0 SH, which is a spin-SH coefficient of scalar multiplication of another spin-2 SH by a spin-0 SH, is written as:

$$\int_{\hat{\mathbb{S}}^2} {}_2Y^*_{l_1 m_1} Y_{l_2 m_2} {}_2Y_{l_3 m_3} \mathrm{d}\hat{\omega} = (-1)^{m_1} \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ -2 & 0 & 2 \end{pmatrix}. \tag{146}$$

We do not explicitly introduce what Wigner 3-j symbols are and how we can compute them. However, note that the above two equations have the same kind of special symbols, which only depend on integer indices. Since existing PRT methods have been used to spin-0 triple product, we can also compute spin-2 SH from their implementation.

5.8 Convolution on Stokes Vectors Fields

In this section, we derive polarized spherical convolution as a rotation equivariant linear operator on Stokes vector fields.

Definition 5.11: Rotation equivariant opeartor

A linear operator $K_{\mathcal{F}} \in \mathcal{L}\left(\mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}\right), \mathcal{F}\left(\hat{\mathbb{S}}^2, \mathcal{S}\right)\right)$ on Stokes vector fields called *rotation equivariant* if $\vec{R}_{\mathcal{F}}\left(K_{\mathcal{F}}\left[\overleftrightarrow{f}\right]\right) = K_{\mathcal{F}}\left[\vec{R}_{\mathcal{F}}\overleftrightarrow{f}\right]$ holds for any $\vec{R} \in \overrightarrow{SO}(3)$ and $\overleftrightarrow{f} : \hat{\mathbb{S}}^2 \to \mathcal{S}_{\hat{\omega}}$.

If such an operator has a kernel, Mueller transform field, then rotation equivariant can be stated as follows.

Proposition 5.12: Rotation equivariant for operator kernel

Suppose there is a Mueller transform field $\overleftrightarrow{K}: \mathbb{S}^2 \times \mathbb{S}^2 \to \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$. The linear operator of the kernel $\overleftrightarrow{K}$, $\overleftrightarrow{K}_{\mathcal{F}}$ is rotation equivariant if and only if $\vec{R}_{\mathcal{F}}\left[\overleftrightarrow{K}\right] = \overleftrightarrow{K}$ for any $\vec{R} \in \overrightarrow{SO}(3)$.

The above condition $\vec{R}_{\mathcal{F}}\left[\overleftrightarrow{K}\right] = \overleftrightarrow{K}$ can be written using the rotation form $\mathbf{K} : \overrightarrow{SO}(3) \times \overrightarrow{SO}(3) \to \mathbb{R}^{4\times 4}$ of the Mueller transform as follows:

$$\mathbf{K}\left(\vec{R}\vec{R}_i, \vec{R}\vec{R}_o\right) = \mathbf{K}\left(\vec{R}_i, \vec{R}_o\right), \quad \forall \vec{R}, \vec{R}_i, \vec{R}_o \in \overrightarrow{SO}(3). \tag{147}$$

Then, we finally obtain a minimal form of the rotation equivariant (operator) kernel. It can be considered an extension that a rotation equivariant operator on scalar fields has been characterized by a simple azimuthal symmetric scalar field. However, we have more information in the codomain of the polarized convolution kernel.

Proposition 5.13: Minimal form of a rotation equivariant operator kernel

A Mueller transform field $\overleftrightarrow{K}: \mathbb{S}^2 \times \mathbb{S}^2 \to \mathcal{M}_{\hat{\omega}_i \to \hat{\omega}_o}$ which is a kernel of rotation equivariant linear operator can be characterized by a Mueller transform function of a single angle as $\mathbf{K}\left(\vec{I}, \vec{R}_{\hat{y}_g}(\beta)\right)$, where $\mathbf{K}$ denotes the rotation form of $\overleftrightarrow{K}$.

Proof: Using rotation equivariance $\mathbf{K}\left(\vec{R}\vec{R}_i, \vec{R}\vec{R}_o\right) = \mathbf{K}\left(\vec{R}_i, \vec{R}_o\right)$, we get:

$$\mathbf{K}\left(\vec{R}_i, \vec{R}_o\right) = \mathbf{K}\left(\vec{I}, \vec{R}_i^{-1}\vec{R}_o\right).$$

With ZYZ Euler angle $\vec{R}_{\hat{z}_g\hat{y}_g\hat{z}_g}(\alpha, \beta, \gamma) = \vec{R}_i^{-1}\vec{R}_o$ and the constraint of rotation forms of Mueller transform fields,

$$\mathbf{K}\left(\vec{I}, \vec{R}_i^{-1}\vec{R}_o\right) = \mathbf{R}_{1:2}(-2\gamma)\,\mathbf{K}\left(\vec{I}, \vec{R}_{\hat{y}_g}(\beta)\right)\mathbf{R}_{1:2}(-2\alpha). \tag{148}$$

□

Note that $\mathbf{K}\left(\vec{I}, \vec{R}_{\hat{y}_g}(\beta)\right) =: \mathbf{k}(\beta) \in \mathbb{R}^{4\times 4}$ is the polarized convolution kernel which also introduced in our main paper. Equation (148) yields its constraints by substituting $\beta = 0$ and $\beta = \pi$ and using $\vec{R}_{zyz}(\alpha, 0, \gamma) = \vec{R}_z(\alpha + \gamma)$ and $\vec{R}_{zyz}(\alpha, \pi, \gamma) = \vec{R}_{zy}(\alpha - \gamma, \pi)$:

$$\begin{aligned} \mathbf{k}(0) &= \mathbf{R}_{1:2}(\psi)\,\mathbf{k}(0)\,\mathbf{R}_{1:2}(-\psi), \\ \mathbf{k}(\pi) &= \mathbf{R}_{1:2}(\psi)\,\mathbf{k}(\pi)\,\mathbf{R}_{1:2}(\psi), \end{aligned} \tag{149}$$

for any ψ. A particular corollary of it is that the isomorphic and conjugation parts of spin 2-to-2 submatrix of $\mathbf{k}$ become zero at $\theta = \pi$ and $\theta = 0$, respectively. These constraints are highly related to each subspace of PSH bases for each submatrix of convolution kernels.

5.9 Convolution in Polarized Spherical Harmonics

Note that the following lemma is useful. It comes from Wigner D-function identities.

Lemma 5.13.1. *For any indices l_i, m_i, and m_i' for $i = 1, 2$ in the valid range and rotation transforms $\vec{S}, \vec{T} \in \overrightarrow{SO}(3)$,*

$$\begin{aligned} \int_{\overrightarrow{SO}(3)} D^{l_1}_{m_1 m_1'}\left(\vec{R}\vec{S}\right) D^{l_2 *}_{m_2 m_2'}\left(\vec{R}\vec{T}\right) \mathrm{d}\mu\left(\vec{R}\right) &= \frac{8\pi^2}{2l_1 + 1}\delta_{(l_1 m_1)(l_2 m_2)} D^{l_1 *}_{m_1' m_2'}\left(\vec{S}^{-1}\vec{T}\right) \\ &= \frac{8\pi^2}{2l_1 + 1}\delta_{(l_1 m_1)(l_2 m_2)} D^{l_1}_{m_2' m_1'}\left(\vec{T}^{-1}\vec{S}\right). \end{aligned} \tag{150}$$

Spin 0-to-2. Starting from the definition, an entry of the coefficient matrix of a rotation equivariant linear operator on Stokes vector fields $\overleftrightarrow{K}_{\mathcal{F}}$ is obtained as follows. Note that in this section, $\tilde{K}$ denotes the complex form of the Mueller transform $\overleftrightarrow{K}$, and $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ denotes the inner product on the Stokes space over scalar $\mathbb{C}$.

$$\begin{aligned} \mathrm{K}_{l_o m_o p_o, l_i m_i 0} &= \left\langle \overleftrightarrow{Y}_{l_o m_o p_o}, \overleftrightarrow{K}_{\mathcal{F}}\left[\mathrm{Y}^R_{l_i m_i}\right]\right\rangle = \iint_{\hat{\mathbb{S}}^2\times\hat{\mathbb{S}}^2} \left\langle \overleftrightarrow{Y}_{l_o m_o p_o}(\hat{\omega}_o), \overleftrightarrow{K}_{\mathrm{p}0}(\hat{\omega}_i, \hat{\omega}_o)\,\mathrm{Y}^R_{l_i m_i}(\hat{\omega}_i)\right\rangle_{\mathcal{S}} \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o = \\ &= \int_0^\pi \int_{\overrightarrow{SO}(3)} \left[\overleftrightarrow{Y}_{l_o m_o p_o}\left(\vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\hat{z}\right)\right]^{\vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\vec{\mathrm{F}}_g} \cdot \left[\overleftrightarrow{K}_{\mathrm{p}0}\left(\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\hat{z}, \vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\hat{z}\right)\mathrm{Y}^R_{l_i m_i}\left(\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\hat{z}\right)\right]^{\vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\vec{\mathrm{F}}_g} \mathrm{d}\vec{R}\sin\theta\mathrm{d}\theta. \end{aligned} \tag{151}$$

Using the relation between spin-weighted spherical harmonics and Wigner-D functions in Definition 5.2,

$$\text{Eq. (151)} = B_{l_i l_o} \Re \left[i^{1-p_o} \int_0^\pi \int_{\overrightarrow{SO}(3)} D^{l_o}_{m_o,-2} \left(\vec{R}\vec{R}_y \left(\frac{\theta}{2}\right)\right) \tilde{K}_{\mathrm{p}0} \left(\vec{R}\vec{R}_y \left(-\frac{\theta}{2}\right), \vec{R}\vec{R}_y \left(\frac{\theta}{2}\right)\right) D^{l_i,R}_{m_i 0} \left(\vec{R}\vec{R}_y \left(-\frac{\theta}{2}\right)\right) \mathrm{d}\vec{R} \sin\theta \mathrm{d}\theta \right]$$
$$=: B_{l_i l_o} \Re \left[i^{1-p_o} I_1 \right], \tag{152}$$

where $B_{l_i l_o} := \frac{\sqrt{(2l_i+1)(2l_i+1)}}{4\pi}$, and we are denoting the integral term as I_1 for simplicity of later steps. In the integrand of I_1, rotation equivariance of $\tilde{K}_{\mathrm{p}0}$ yields:

$$\tilde{K}_{\mathrm{p}0} \left(\vec{R}\vec{R}_y \left(-\frac{\theta}{2}\right), \vec{R}\vec{R}_y \left(\frac{\theta}{2}\right)\right) = \tilde{K}_{\mathrm{p}0} \left(\vec{I}, \vec{R}_y (\theta) \right). \tag{153}$$

We observe that it is independent of $\vec{R}$ so that it can go outside of the integral over $\vec{R}$. Then, using the relation between real and complex Wigner-D functions in Equation (60), I_1 becomes:

$$I_1 = \int_0^\pi \tilde{K}_{\mathrm{p}0} \left(\vec{I}, \vec{R}_y (\theta) \right) \int_{\overrightarrow{SO}(3)} D^{l_o}_{m_o,-2} \left(\vec{R}\vec{R}_y \left(\frac{\theta}{2}\right)\right) \sum_{m_c = \pm m_i} \left(M^{C\to R}_{m_i m_c} \right)^* D^{l_i}_{m_c 0} \left(\vec{R}\vec{R}_y \left(-\frac{\theta}{2}\right)\right) \mathrm{d}\vec{R} \sin\theta \mathrm{d}\theta. \tag{154}$$

Then we can use Lemma 5.13.1 to the inner integral with a symmetry of Wigner-D functions $D^{l_i}_{m_c 0} = (-1)^{m_c} D^{l_i,*}_{-m_c 0}$.

$$I_1 = \delta_{l_i l_o} \frac{2\pi}{B_{l_i l_i}} \int_0^\pi \tilde{K}_{\mathrm{p}0} \left(\vec{I}, \vec{R}_y (\theta) \right) \sum_{m_c = \pm m_i} \delta_{m_o, -m_c} (-1)^{m_c} \left(M^{C\to R}_{m_i m_c} \right)^* D^{l_i}_{0,-2} \left(\vec{R}_y (\theta) \right) \sin\theta \mathrm{d}\theta. \tag{155}$$

Here, we observe that $I_1 = 0$ if $|m_i| = |m_o|$, terms containing m_i, m_o, and m_s, denoted by $\mathbf{U}^{\mathrm{p}0}$, is evaluated as follows.

$$\mathbf{U}^{\mathrm{p}0} := \mathbf{Mat} \left[\sum_{m_c = \pm m_i} \delta_{m_o, -m_c} (-1)^{m_c} \left(M^{C\to R}_{m_i m_c} \right)^* \mid m_o, m_i = +|m|, -|m| \right]$$
$$= (-1)^m \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ (-1)^m & -(-1)^m i \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ (-1)^m & (-1)^m i \end{bmatrix}. \tag{156}$$

Then, combining Equations (155) and (156) and converting the Wigner-D to a spin-weighted spherical harmonics conversely, we get

$$I_1 = \delta_{(l_i|m_i|)(l_o|m_o|)} U^{\mathrm{p}0}_{m_o m_i} \frac{2\pi}{B_{l_i l_i l_i}} \int_0^{2\pi} \left\langle \overset{\leftrightarrow}{Y}_{l0} (\theta, 0), \overset{\leftrightarrow}{K}_{\mathrm{p}0} \left(\hat{z}_g, \hat{\omega}_{\mathrm{sph}} (\theta, 0) \right) \right\rangle_{\mathbb{C}} \sin\theta \mathrm{d}\theta. \tag{157}$$

We observe here that, similar to conventional convolution through scalar spherical harmonics, the only degree of freedom comes from the order l. Thus, we can define the *convolution coefficient* of the scalar-to-Stokes part of $\overset{\leftrightarrow}{K}$ as follows.

$$\mathrm{k}_{l,\mathrm{p}0} = 2\pi \int_0^\pi \left\langle \overset{\leftrightarrow}{Y}_{l0p} (\theta, 0), \overset{\leftrightarrow}{K}_{\mathrm{p}0} \left(\hat{z}_g, \hat{\omega}_{\mathrm{sph}} (\theta, 0) \right) \right\rangle_{\mathbb{C}} \sin\theta \mathrm{d}\theta \in \mathbb{C}, \tag{158}$$

which can be considered as an inner product over the entire $\hat{\mathbb{S}}^2$ as scalar SH convolution is. Note that we are defining the convolution coefficient $\mathrm{k}_{l,\mathrm{p}0}$ as a complex number so that we will take $\Re$ in later steps.

Now we finally get the coefficient of the linear operator by combining Equations (152), (157), and (158).

$$\mathrm{K}_{l_o m_o p_o, l_i m_i 0} = \delta_{(l_i|m_i|)(l_o|m_o|)} \sqrt{\frac{4\pi}{2l_i+1}} \Re \left(i^{1-p_o} U^{\mathrm{p}0}_{m_o m_i} \mathrm{k}_{l,\mathrm{p}0} \right), \tag{159}$$

where $p_o = 1, 2$.

Spin 2-to-0. We can follow similar steps to scalar-Stokes components. The coefficient of the linear operator is:

$$\begin{aligned}
&\mathrm{K}_{l_o m_o 0, l_i m_i p_i} = \left\langle Y^R_{l_o m_o}, \overleftrightarrow{K}_{\mathcal{F}}\left[\overleftrightarrow{Y}_{l_i m_i p_i}\right]\right\rangle = \iint_{\hat{\mathbb{S}}^2\times\hat{\mathbb{S}}^2} Y^R_{l_o m_o}(\hat{\omega}_o)\left\langle \overleftrightarrow{K}_{0\mathbf{p}}(\hat{\omega}_i,\hat{\omega}_o), \overleftrightarrow{Y}_{l_i m_i p_i}(\hat{\omega}_i)\right\rangle_{\mathcal{S}} \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o = \\
&= \int_0^\pi\int_{\overrightarrow{SO}(3)} Y^R_{l_o m_o}\left(\vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\hat{z}\right)\left[\overleftrightarrow{K}_{0\mathbf{p}}\left(\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\hat{z}, \vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\hat{z}\right)\right]^{\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\vec{\mathbf{F}}_g} \cdot \left[\overleftrightarrow{Y}_{l_i m_i p_i}\left(\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\hat{z}\right)\right]^{\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\vec{\mathbf{F}}_g} \mathrm{d}\vec{R}\sin\theta \mathrm{d}\theta \\
&= B_{l_i l_o}\Re\left[i^{p_i-1}\int_0^\pi\int_{\overrightarrow{SO}(3)} D^{l_o,R}_{m_o,0}\left(\vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\right)\tilde{K}_{0\mathbf{p}}\left(\vec{I},\vec{R}_y(\theta)\right)D^{l_i,*}_{m_i,-2}\left(\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\right)\mathrm{d}\vec{R}\sin\theta\mathrm{d}\theta\right] =: B_{l_i l_o}\Re\left[p^{p_i-1}I_2\right].
\end{aligned} \tag{160}$$

Here, we denote the integral term by I_2. Using the relation between real and complex Wigner-D functions in Equation (60) followed by Lemma 5.13.1,

$$\begin{aligned}
I_2 &= \int_0^\pi \tilde{K}_{0\mathbf{p}}\left(\vec{I},\vec{R}_y(\theta)\right)\int_{\overrightarrow{SO}(3)}\sum_{m_c=\pm m_o}\left(M^{C\to R}_{m_o m_c}\right)^* D^{l_o}_{m_c,0}\left(\vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)\right)D^{l_i,*}_{m_i,-2}\left(\vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)\right)\mathrm{d}\vec{R}\sin\theta\mathrm{d}\theta \\
&= \delta_{l_i l_o}\frac{2\pi}{B_{l_i l_i}}\int_0^\pi \tilde{K}_{0\mathbf{p}}\left(\vec{I},\vec{R}_y(\theta)\right)\sum_{m_c=\pm m_o}\delta_{m_c m_i}\left(M^{C\to R}_{m_o m_c}\right)^* D^{l_i}_{-2,0}\left(\vec{R}_y(\theta)\right)\sin\theta\mathrm{d}\theta.
\end{aligned} \tag{161}$$

On the right-hand side, we can reduce the terms containing m_c, denoted by $\mathbf{U}^{0\mathbf{p}}$ as follows:

$$\begin{aligned}
\mathbf{U}^{0\mathbf{p}} &:= \mathbf{Mat}\left[\sum_{m_c=\pm m_o}\delta_{m_c m_i}\left(M^{C\to R}_{m_o m_c}\right)^* \mid m_o, m_i = +|m|, -|m|\right] \\
&= \mathbf{Mat}\left[\left(M^{C\to R}_{m_o m_i}\right)^* \mid m_o, m_i = +|m|, -|m|\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & (-1)^m \\ i & -(-1)^m i\end{bmatrix}.
\end{aligned} \tag{162}$$

Combining Equations (161) and (162) yields:

$$I_2 = \delta_{(l_i|m_i|)(l_o|m_o|)}U^{0\mathbf{p}}_{m_o m_i}\frac{2\pi}{B_{l_i l_i l_i}}\int_0^\pi Y^{C,*}_{l,-2}(\theta,0)\left\langle\overleftrightarrow{K}_{0\mathbf{p}}\left(\hat{z}_g,\hat{\omega}_{\mathrm{sph}}(\theta,0)\right), \begin{bmatrix}1\\0\end{bmatrix}_{\vec{\mathbf{F}}_g}\right\rangle_{\mathbb{C}}\sin\theta\mathrm{d}\theta. \tag{163}$$

Now we can finally define the convolution coefficient of the Stokes-to-scalar part of $\overleftrightarrow{K}$, denoted by $\mathrm{k}_{l,0\mathbf{p}}$, and obtain the coefficient of a linear operator in terms of $\mathrm{k}_{l,0\mathbf{p}}$.

$$\mathrm{k}_{l,0\mathbf{p}} = 2\pi\int_0^\pi Y^{C,*}_{l,-2}(\theta,0)\left\langle\overleftrightarrow{K}_{0\mathbf{p}}\left(\hat{z}_g,\hat{\omega}_{\mathrm{sph}}(\theta,0)\right), \begin{bmatrix}1\\0\end{bmatrix}_{\vec{\mathbf{F}}_g}\right\rangle_{\mathbb{C}}\sin\theta\mathrm{d}\theta \in \mathbb{C}, \tag{164}$$

$$\mathrm{K}_{l_o m_o 0, l_i m_i p_i} = \delta_{(l_i|m_i|)(l_o|m_o|)}\sqrt{\frac{4\pi}{2l_i+1}}\Re\left(i^{p_i-1}U^{0\mathbf{p}}_{m_o m_i}\mathrm{k}_{l,0\mathbf{p}}\right), \tag{165}$$

where $p_o = 1, 2$.

Spin 2-to-2. The coefficient of the linear operator is:

$$\mathrm{K}_{l_o m_o p_o, l_i m_i p_i} = \left\langle \overleftrightarrow{Y}_{l_o m_o p_o}, \overleftrightarrow{K}_{\mathcal{F}} \left[\overleftrightarrow{Y}_{l_i m_i p_i} \right] \right\rangle = \iint_{\hat{\mathbb{S}}^2 \times \hat{\mathbb{S}}^2} \left\langle \overleftrightarrow{Y}_{l_o m_o p_o} \left(\hat{\omega}_o \right), \overleftrightarrow{K}_{\mathbf{pp}} \left(\hat{\omega}_i, \hat{\omega}_o \right) \left[\overleftrightarrow{Y}_{l_i m_i p_i} \left(\hat{\omega}_i \right) \right] \right\rangle_{\mathcal{S}} \mathrm{d}\hat{\omega}_i \mathrm{d}\hat{\omega}_o \tag{166}$$

$$= \int_0^\pi \int_{\overrightarrow{SO}(3)} \left[\overleftrightarrow{Y}_{l_o m_o p_o} \left(\vec{R}_o \hat{z} \right) \right]^{\vec{\mathbf{F}}_o} \cdot \left[\overleftrightarrow{K}_{\mathbf{pp}} \left(\vec{R}_i \hat{z}, \vec{R}_o \hat{z} \right) \right]^{\vec{\mathbf{F}}_i \to \vec{\mathbf{F}}_o} \left[\overleftrightarrow{Y}_{l_i m_i p_i} \left(\vec{R}_i \hat{z} \right) \right]^{\vec{\mathbf{F}}_o} \mathrm{d}\vec{R} \sin\theta \mathrm{d}\theta, \tag{167}$$

where $\vec{R}_i := \vec{R}\vec{R}_y\left(-\frac{\theta}{2}\right)$, $\vec{R}_o := \vec{R}\vec{R}_y\left(\frac{\theta}{2}\right)$, $\vec{\mathbf{F}}_i := \vec{R}_i\vec{\mathbf{F}}_g$, and $\vec{\mathbf{F}}_o := \vec{R}_o\vec{\mathbf{F}}_g$. It can be rewritten in terms of Wigner-D functions as:

$$\mathrm{K}_{l_o m_o p_o, l_i m_i p_i} = B_{l_i l_o} \int_0^\pi \int_{\overrightarrow{SO}(3)} \mathbb{R}^2 \left(i^{p_o - 1} D^{l_o,*}_{m_o,-2} \left(\vec{R}_o \right) \right)^T \tilde{\mathbf{K}}_{\mathbf{pp}} \left(\vec{R}_i, \vec{R}_o \right) \mathbb{R}^2 \left(i^{p_i - 1} D^{l_i,*}_{m_i,-2} \left(\vec{R}_i \right) \right) \mathrm{d}\vec{R} \sin\theta \mathrm{d}\theta. \tag{168}$$

Here, we need an additional step that was not needed for scalar-to-Stokes and Stokes-to-scalar terms. Note that 2×2 matrix $\tilde{\mathbf{K}}$ can be decompose into two terms $\tilde{\mathbf{K}}_{\mathbf{pp}} = \mathbb{R}^{2\times2}\left(\tilde{K}_{\mathrm{ppi}}\right) + \mathbb{R}^{2\times2}\left(\tilde{K}_{\mathrm{ppc}}\right)\mathbf{J}$. Then right two terms in the integral in Equation (168) become as follows by Equation (114):

$$\tilde{\mathbf{K}}_{\mathbf{pp}} \left(\vec{R}_i, \vec{R}_o \right) \mathbb{R}^2 \left(i^{p_i - 1} D^{l_i,*}_{m_i,-2} \left(\vec{R}_i \right) \right) = \mathbb{R}^2 \left(i^{p_i - 1} \tilde{K}_{\mathrm{ppi}} \left(\vec{R}_i, \vec{R}_o \right) D^{l_i,*}_{m_i,-2} \left(\vec{R}_i \right) + i^{1 - p_i} \tilde{K}_{\mathrm{ppc}} \left(\vec{R}_i, \vec{R}_o \right) D^{l_i}_{m_i,-2} \left(\vec{R}_i \right) \right). \tag{169}$$

Substituting this results into Equation (168),

$$\begin{aligned}
\mathrm{K}_{l_o m_o p_o, l_i m_i p_i} &= B_{l_i l_o} \Re \left[i^{p_i - p_o} \int_0^\pi \int_{\overrightarrow{SO}(3)} D^{l_o}_{m_o,-2} \left(\vec{R}_o \right) \tilde{K}_{\mathrm{ppi}} \left(\vec{R}_i, \vec{R}_o \right) D^{l_i,*}_{m_i,-2} \left(\vec{R}_i \right) \mathrm{d}\vec{R} \sin\theta \mathrm{d}\theta \right] \\
&+ B_{l_i l_o} \Re \left[i^{2 - p_i - p_o} \int_0^\pi \int_{\overrightarrow{SO}(3)} D^{l_o}_{m_o,-2} \left(\vec{R}_o \right) \tilde{K}_{\mathrm{ppc}} \left(\vec{R}_i, \vec{R}_o \right) D^{l_i}_{m_i,-2} \left(\vec{R}_i \right) \mathrm{d}\vec{R} \sin\theta \mathrm{d}\theta \right] \\
&=: B_{l_i l_o} \Re \left[i^{p_i - p_o} I_3 + i^{2 - p_i - p_o} I_4 \right].
\end{aligned} \tag{170}$$

Here, we denote two integral terms by I_3 and I_4, respectively. First, I_3 can be evaluated similarly to previous components.

$$I_3 = \delta_{(l_i m_i)(l_o m_o)} \frac{2\pi}{B_{l_i l_i}} \int_0^\pi \tilde{K}_{\mathrm{ppi}} \left(\vec{I}, \vec{R}_y(\theta) \right) D^{l_i}_{-2,-2} \left(\vec{R}_y(\theta) \right) \sin\theta \mathrm{d}\theta \tag{171}$$

$$= \delta_{(l_i m_i)(l_o m_o)} \frac{2\pi}{B_{l_i l_i l_i}} \int_0^\pi \left\langle {}_2Y_{l_i,-2}(\theta, 0), \tilde{K}_{\mathrm{ppi}} \left(\vec{I}, \vec{R}_y(\theta) \right) \right\rangle_{\mathbb{C}} \sin\theta \mathrm{d}\theta. \tag{172}$$

Similarly, I_4 is:

$$I_4 = \delta_{(l_i,-m_i)(l_o m_o)} (-1)^{m_i} \frac{2\pi}{B_{l_i l_i}} \int_0^\pi \tilde{K}_{\mathrm{ppc}} \left(\vec{I}, \vec{R}_y(\theta) \right) D^{l_i}_{2,-2} \left(\vec{R}_y(\theta) \right) \sin\theta \mathrm{d}\theta \tag{173}$$

$$= \delta_{(l_i,-m_i)(l_o m_o)} (-1)^{m_i} \frac{2\pi}{B_{l_i l_i l_i}} \int_0^\pi \left\langle {}_2Y_{l_i 2}(\theta, 0), \tilde{K}_{\mathrm{ppc}} \left(\vec{I}, \vec{R}_y(\theta) \right) \right\rangle_{\mathbb{C}} \sin\theta \mathrm{d}\theta. \tag{174}$$

Finally, the convolution coefficient of the Stokes-to-Stokes part of $\overset{\leftrightarrow}{K}$ can be defined as two complex numbers $\mathrm{k}_{l,\mathrm{pp}a}$ and $\mathrm{k}_{l,\mathrm{pp}b}$, and the coefficient of linear operator can be written in terms of $\mathrm{k}_{l,\mathrm{pp}a}$ and $\mathrm{k}_{l,\mathrm{pp}b}$.

$$\mathrm{k}_{l,\mathrm{ppi}} = 2\pi \int_0^{\pi} \left\langle {}_2Y_{l_i,-2}\left(\theta,0\right), \tilde{K}_{\mathrm{ppi}}\left(\vec{I}, \vec{R}_y\left(\theta\right)\right)\right\rangle_{\mathbb{C}} \sin\theta \mathrm{d}\theta \in \mathbb{C}, \tag{175}$$

$$\mathrm{k}_{l,\mathrm{ppc}} = 2\pi \int_0^{\pi} \left\langle {}_2Y_{l_i,2}\left(\theta,0\right), \tilde{K}_{\mathrm{ppc}}\left(\vec{I}, \vec{R}_y\left(\theta\right)\right)\right\rangle_{\mathbb{C}} \sin\theta \mathrm{d}\theta \in \mathbb{C}, \tag{176}$$

$$\mathrm{K}_{l_o m_o p_o, l_i m_i p_i} = \delta_{l_i l_o}\sqrt{\frac{4\pi}{2l_i+1}}\,\Re\left(\delta_{m_i m_o} i^{p_i - p_o}\mathrm{k}_{l,\mathrm{ppi}} + \delta_{-m_i m_o}\left(-1\right)^{m_i} i^{2-p_i-p_o}\mathrm{k}_{l,\mathrm{ppc}}\right), \tag{177}$$

where $p_o = 1, 2$. Note that the final equation can be rewritten using Equations (118) and (119):

$$\mathbf{Mat}\left[\mathrm{K}_{l_o m_o p_o, l_i m_i p_i} \mid p_o, p_i\right] = \delta_{l_i l_o}\sqrt{\frac{4\pi}{2l_i+1}}\left(\delta_{m_i m_o}\mathbb{R}^{2\times 2}\left(\mathrm{k}_{l,\mathrm{ppi}}\right) + \delta_{-m_i m_o}\left(-1\right)^{m_i}\mathbb{R}^{2\times 2}\left(\mathrm{k}_{l,\mathrm{ppc}}\right)\mathbf{J}\right). \tag{178}$$

We observe that this expression is natural since $\tilde{K}_{\mathrm{ppi}}$ and $\tilde{K}_{\mathrm{ppc}}$ was decomposed from $\tilde{\mathbf{K}}_{\mathbf{pp}} = \mathbb{R}^{2\times 2}\left(\tilde{K}_{\mathrm{ppi}}\right) + \mathbb{R}^{2\times 2}\left(\tilde{K}_{\mathrm{ppc}}\right)\mathbf{J}$.

6 RESULTS AND DISCUSSION

6.1 Results for Precomputed Polarized Radiance Transfer

Scene specification. We provide technical details of the scene setups throughout the main paper (Figures 1, 20 20, 21, 18, and 22) and this supplemental document (Figure 9) in Table 2. The reported numbers of vertices include 3D models themselves and ground planes. Note that While (1) lighting (environment map), (2) radiance transfer matrix of pBRDF and shadow, and (3) high-order convolution approximation of pBRDF are encoded in PSH coefficients, each single coefficient contains trichromatic RGB values, refer to 12 bytes (4x3 bytes float). Each scene uses two materials. Note that while transfer matrices differ for each vertex, convolution coefficients for high-order pBRDF are shared by all vertices of the same material due to rotation equivariance.

Validation against GT. Here we provide rendered images of Mitsuba 3 GT render and our PPRT method for each cut-off frequency $l_{\max}$, which are discussed in Figure 19 in the main paper Section 7. The resulting images and difference maps are shown in Figure 9. Since Mitsuba 3 does not support polarized environment map emitters, we are using an unpolarized environment map for this scene. In addition, Baek et al. [2020]'s data-based pBRDFs are only supported by multispectral variants of Mitsuba 3, while our implementation is based on conventional RGB rendering by projecting multi-channel Baek et al. [2020] pBRDF into RGB in advance. Instead, for this quantitative validation, we conducted this scene with an analytic pBRDF model Baek et al. [2018]. Specific configurations of this scene are also reported in Table 2.

Table 2. The scene setups specification throughout the main paper and this supplemental document. For several scenes which do not use high-order convolution approximation we are not reporting numbers of such coefficients.

Scene		# of vertices	Lighting coeff.	Radiance transfer matrix (per vertex)	Convolution coeff. (per material)	FPS
Main Fig. 1		21,087	300	5,625	45	100
Main Fig. 20(a)	Rows 1 & 2	10,115	75	5,625	–	475
	Row 3		300	5,625	45	306
Main Fig. 20(b)	Rows 1 & 2	20,545	75	5,625	–	162
	Row 3		300	5,625	45	102
Main Fig. 21	(b)	10,115	75	5,625	–	480
	(c)		108	11,664	–	210
	(d)		300	5,625	45	308
Main Figs. 18, 22		19,944	300	5,625	45	111
Fig. 9	$l_{\max} = 4$	3,482	75	5,625	–	750
	$l_{\max} = 5$		108	11,664	–	373
	$l_{\max} = 6$		147	21,609	–	208
	$l_{\max} = 7$		192	36,864	–	110
	$l_{\max} = 8$		243	59,049	–	75

6.2 Discussion on SWSH Formulations in Previous Work

Definitions of spin-weight spherical harmonics. For interested readers, we briefly review the formulations of SWSH in previous work here. When SWSH were originally introduced by Newman and Penrose [1966], they were defined using a special kind of differential operators, spin raising and lowering operators $\eth$ and $\bar{\eth}$. Then Newman and Penrose [1966] defined SWSH in the spherical coordinates (θ, ϕ), and dependency of local frames is regarded implicitly. Goldberg et al. [1967] found a relationship between SWSH and Wigner D-functions. Our

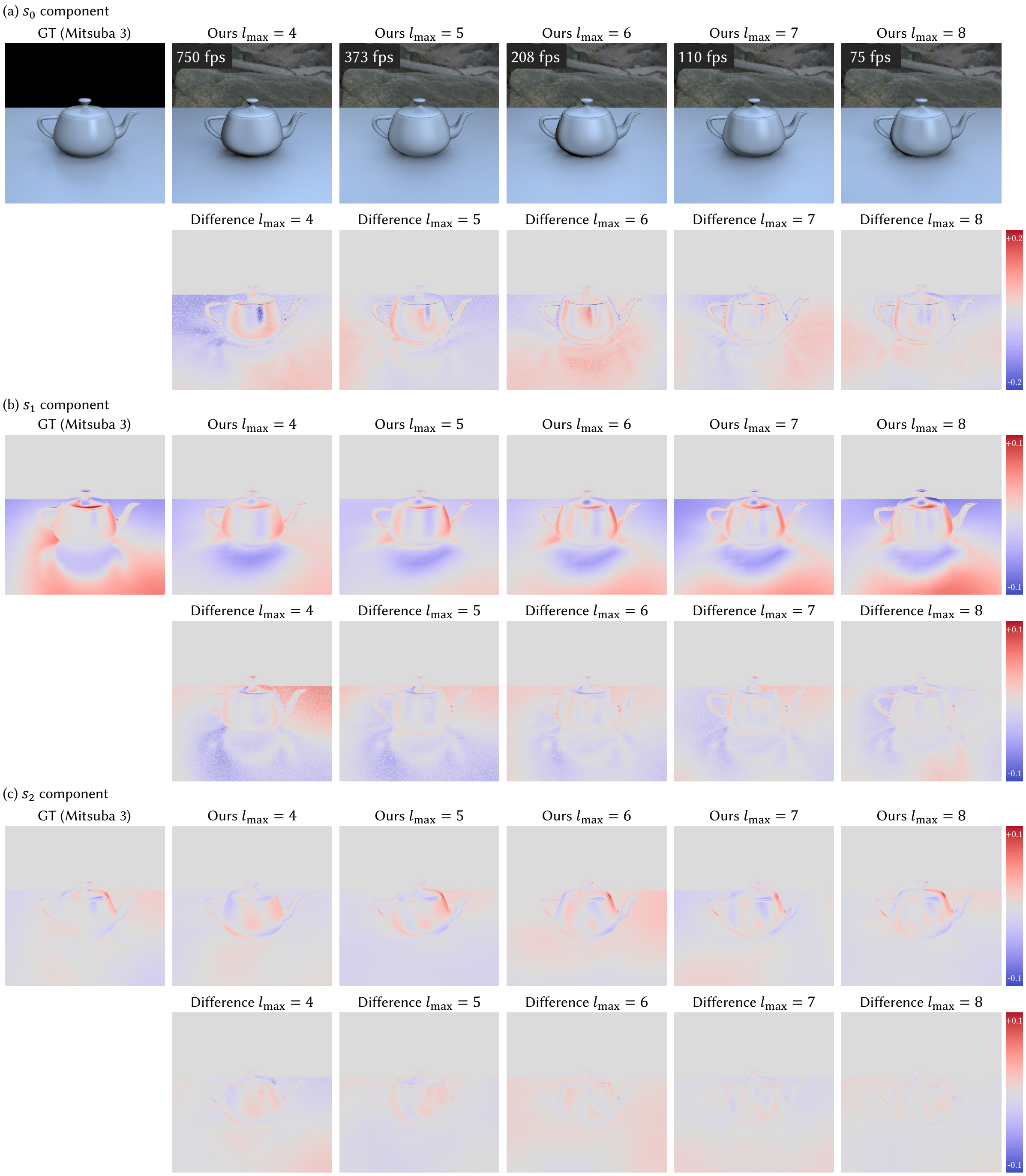

Fig. 9. Rendered images for Figure 19 in the main paper. We validate our real-time polarized rendering with shadowed radiance transfer compared with Mitsuba 3 GT ray tracer. (a) to (c) shows s_0, s_1, and s_2 Stokes components of polarized images, respectively. We observe our results get closer to GT results as the cut-off frequency $l_{\max}$ increases. Note that differences in this figure and Figure 19 in the main paper were computed only at pixels where the object exists.

description of SWSH in Definition 5.2 is based on this relationship to make the frame dependency clear rather than implicit. Note that we do not cover what ð and $\bar{ð}$ operators are.

Spin-weight $s = -2$ spherical harmonics. Ng and Liu [1999]; Zaldarriaga and Seljak [1997] used both spin +2 and −2 SH to handle the correlation of Stokes vector fields, but the necessity of two types of special functions for describing a single type of quantity, Stokes vectors, has been somewhat counterintuitive. While the occurrence of complex conjugation in several equations in this work (${}_2Y^*_{l_i m_i}$ and $\tilde{\mathrm{f}}^*_{l,-mp}$ in Equations (51c) and (69b) in the main paper, respectively) can be considered to correspond to the spin −2 coefficients in Ng and Liu [1999]; Zaldarriaga and Seljak [1997], we do not need to introduce spin −2 SH in our paper. Instead, the complex conjugation is explained not as a property of special functions such as spin ±2 SH, but by the *complex pair separation* of Mueller transform (Equation (47a) in the main paper and Equation (112) in this supplemental document), which is defined in the angular domain without regarding any basis function.

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