



CS482: Interactive Computer Graphics

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LINEAR TRANSFORMATION

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Point vs. Vector



- Represent Points using coordinates
- To perform geometric transformations to these points
- Vectors: 3D motion via linear transformations
- Coordinate vector: the position of the point

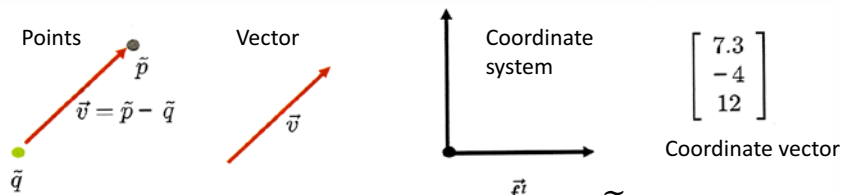
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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Point vs. Coordinate Vector



1. Point (geometric object): notated as \tilde{p} (tilde above the letter), non-numerical object.
2. Vector (motion): notated as \vec{v} (arrow above the letter), non-numerical object.
3. Coordinate system: denoted as \vec{f}^t (bold: column vector, t makes it transpose), non-numerical object basis for vector; frame for point
4. **Coordinate vector**: noted as \mathbf{c} (bold letter), **numerical** object

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Geometric data types



- Point: \tilde{p}
 - represents place
- Vector: \vec{v}
 - represents motion/offset between points
- Coordinate vector: \mathbf{c}
- Coordinate system: \vec{s}
 - “basis” for vectors
 - “frame” for points

Vector Space



- A vector space V is some set of elements \vec{v}
- **NB: Vector (motion) is NOT just a set of three numbers!!!**
- If a set of vectors is not linearly dependent, we call linearly independent.
- If $\vec{b}_1 \dots \vec{b}_n$ are linearly independent, all vectors \vec{v} of V can be expressed with coordinates c_i of a basis of V (a set of \vec{b}_i).

$$\vec{v} = \sum_i c_i \vec{b}_i .$$

- n is the dimension of the basis/space

Vector Space



- Free motion in space, 3 dimensional vector
- In vector algebra notation:

$$\vec{v} = \sum_i c_i \vec{b}_i = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

- a vector \vec{v}
 - row basis vectors \vec{b}^t
 - column coordinate vector \mathbf{c}
- $$\vec{v} = \vec{b}^t \mathbf{c}.$$

Linear Transformation



- Linear transformation follows these two properties:

$$L(\vec{v} + \vec{u}) = L(\vec{v}) + L(\vec{u})$$

$$L(\alpha \vec{v}) = \alpha L(\vec{v}).$$

- Vector transformation (such that the basis is linear independent):

$$\vec{v} \Rightarrow L(\vec{v}) = L\left(\sum_i c_i \vec{b}_i\right) = \sum_i c_i L(\vec{b}_i).$$

3-by-3 Transformation



- Rewrite the linear transform

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} L(\vec{b}_1) & L(\vec{b}_2) & L(\vec{b}_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

- $L(\vec{b}_1)$ is actually a linear combination of the original basis vectors.

$$L(\vec{b}_1) = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} \\ M_{2,1} \\ M_{3,1} \end{bmatrix}$$

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3-by-3 Transformation



- 3-by-3 matrix:

$$\begin{bmatrix} L(\vec{b}_1) & L(\vec{b}_2) & L(\vec{b}_3) \end{bmatrix} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix}.$$

- Putting all together:

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

- A matrix to transform one vector to another:

$$\vec{v} = \vec{b}^t \mathbf{c} \Rightarrow \vec{b}^t M \mathbf{c}$$

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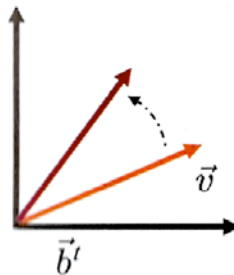
Linear transform of a vector



- A vector undergoes a linear transformation

$$\vec{v} = \vec{b}^t \mathbf{c} \Rightarrow \vec{b}^t M \mathbf{c}$$

- The matrix M depends on the chosen linear transformation.



$$\vec{v} \Rightarrow L(\vec{v})$$

Inverse transform



- Identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

$$MM^{-1} = M^{-1}M = I .$$

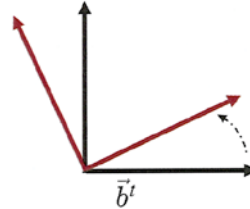
- In 3D graphics, while moving objects around in space, it will seldom make sense to use a non-invertible transform.

Linear transform of a basis



- A basis undergoes a linear transformation

$$\vec{\mathbf{b}}^t \Rightarrow \vec{\mathbf{b}}^t M$$



- Valid to multiply a matrix times a coordinate vector
- change a basis of a vector $\vec{\mathbf{b}}^t$ to $\vec{\mathbf{a}}^t$

$$\vec{\mathbf{a}}^t = \vec{\mathbf{b}}^t M, \quad \vec{\mathbf{v}} = \vec{\mathbf{b}}^t \mathbf{c} = \vec{\mathbf{a}}^t M^{-1} \mathbf{c}.$$

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Dot product



- Input: two vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}}$
- Output: a real number $\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$
- dot product = the squared length $\|\vec{\mathbf{v}}\|^2 := \vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$
- The angle between the two vectors: $\theta \in [0.. \pi]$

$$\cos \theta = \frac{\vec{\mathbf{v}} \cdot \vec{\mathbf{w}}}{\|\vec{\mathbf{v}}\| \|\vec{\mathbf{w}}\|}.$$

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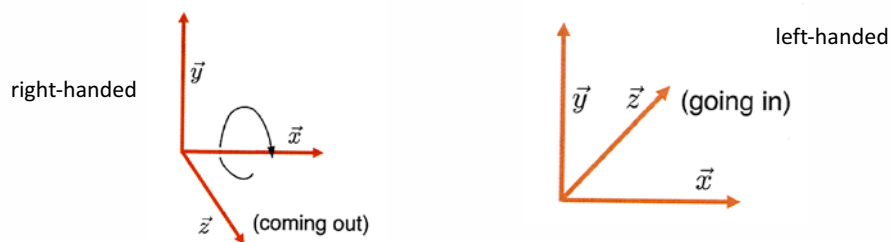
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3D orthogonal basis



- Orthogonal vectors: $\vec{v} \cdot \vec{w} = 0$
- A right-handed orthogonal coordinate system. The z axis comes out of the screen (OpenGL).
- A left-handed orthogonal coordinate system. The z axis goes into the screen (DirectX).



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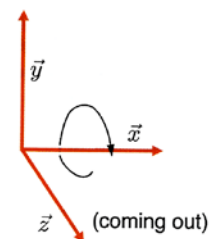
Cross product



- Input: two vectors
- Output: a vector

$$\vec{v} \times \vec{w} := \|\vec{v}\| \|\vec{w}\| \sin \theta \vec{n},$$

- where \vec{n} is a unit vector that is orthogonal to the plane spanned by \vec{v} and \vec{w}
- $[\vec{v}, \vec{w}, \vec{n}]$ forms a right-handed basis



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Cross product



- In a right-handed orthogonal basis $\vec{\mathbf{b}}^t$
- We can compute a cross-product as

$$(\vec{\mathbf{b}}^t \mathbf{c}) \times (\vec{\mathbf{b}}^t \mathbf{d}) = \begin{bmatrix} c_2 d_3 - c_3 d_2 \\ c_3 d_1 - c_1 d_3 \\ c_1 d_2 - c_2 d_1 \end{bmatrix}$$

2D Rotation



- Let $\vec{\mathbf{b}}^t$ be a 2D right-handed orthonormal basis

$$\vec{\mathbf{v}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

- Rotated vector

$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta.$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

3D Rotation



- Every rotation fixes an axis of rotation and rotates by some angle about that axis.
- Rotation around the z axis:

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ \Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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3D Rotation



- Rotation around the x axis

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Rotation around the y axis

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

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xyz-Euler angle rotation



- Axis of rotation $\vec{k} = [k_x, k_y, k_z]^t$

- xyz-Euler angle rotation matrix

$$\begin{bmatrix} k_x^2 v + c & k_x k_y v - k_z s & k_x k_z v + k_y s \\ k_y k_x v + k_z s & k_y^2 v + c & k_y k_z v - k_x s \\ k_z k_x v - k_y s & k_z k_y v + k_x s & k_z^2 v + c \end{bmatrix}$$



where $c := \cos\theta$, $s := \sin\theta$, $v := 1 - c$.

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Scales



- Scaling operations

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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