CS380: Introduction to Computer Graphics
Respect
Chapter 4

Min H. Kim
KAIST School of Computing

Affine Transformation

SUMMARY

Linear transformation

- 3-by-3 transform matrix → 4-by-4 affine transform

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} a & b & c & 0 \\ e & f & g & 0 \\ i & j & k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}.$$

Translation transformation

- translation transformation to points

$$\begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 & \tilde{o} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix}.$$

Affine transform matrix

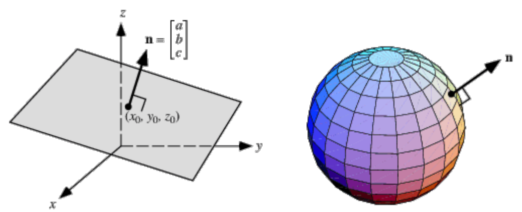
- An affine matrix can be factored into a linear part and a translational part:

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & d \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c & 0 \\ e & f & g & 0 \\ i & j & k & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} l & t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} i & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} l & 0 \\ 0 & 1 \end{bmatrix} \qquad A = TL$$

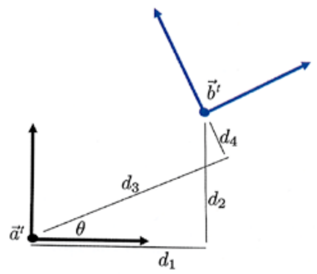
Normals

- Normal: a vector that is orthogonal to the tangent plane of the surfaces at that point.
  - the tangent plane is a plane of vectors that are defined by subtracting (infinitesimally) nearby surface points: $\vec{n} \cdot (\tilde{p}_1 - \tilde{p}_2) = 0$



Chapter 4

RESPECT



Scaling a point over frame

- We are transforming a point $\tilde{p}$ in a frame $\vec{\mathbf{f}}^t$

$$\tilde{p} = \vec{\mathbf{f}}^t \mathbf{c}$$

- With a matrix

$$\mathbf{S} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

the stretches by factor of two in first axis of $\vec{\mathbf{f}}^t$

- Performing a transform: $\vec{\mathbf{f}}^t \mathbf{c} \Rightarrow \vec{\mathbf{f}}^t S \mathbf{c}$
- Suppose another frame: $\vec{\mathbf{a}}^t = \vec{\mathbf{f}}^t A$

Scaling a point over frame

- We could express the point with a new coordinate vector

$$\tilde{p} = \vec{\mathbf{f}}^t\mathbf{c} = \vec{\mathbf{a}}^t\mathbf{d} \qquad \color{blue}{\vec{\mathbf{a}}^t = \vec{\mathbf{f}}^t A}$$
$$\vec{\mathbf{f}}^t\mathbf{c} = \vec{\mathbf{f}}^t A\mathbf{d} \qquad \color{blue}{\vec{\mathbf{f}}^t = \vec{\mathbf{a}}^t A^{-1}}$$
$$\mathbf{d} = A^{-1}\mathbf{c}$$

- Now S transforms the point $\tilde{p}$ with respect to $\vec{\mathbf{a}}^t$

$$\vec{\mathbf{a}}^t\mathbf{d} \Rightarrow \vec{\mathbf{a}}^t S\mathbf{d}$$

Left-of rule

- Point is transformed with respect to the the frame that appears immediately to the left of the transformation matrix in the expression.
- We read

$$\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t S$$

$\vec{\mathbf{f}}^t$ is transformed by S with respect to $\vec{\mathbf{f}}^t$

- We read

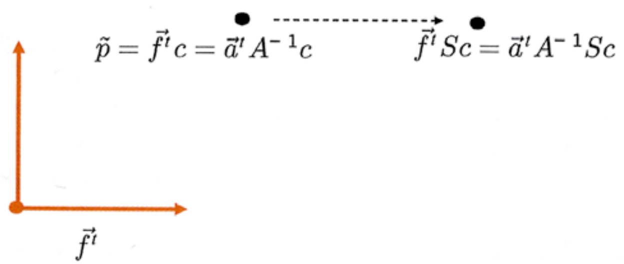
$$\vec{\mathbf{f}}^t = \vec{\mathbf{a}}^t A^{-1} \Rightarrow \vec{\mathbf{a}}^t S A^{-1}$$

$\vec{\mathbf{f}}^t$ is transformed by S with respect to $\vec{\mathbf{a}}^t$

Scaling a point over frame

$$\tilde{p} = \vec{\mathbf{f}}^t\mathbf{c} \Rightarrow \vec{\mathbf{f}}^t S\mathbf{c}$$

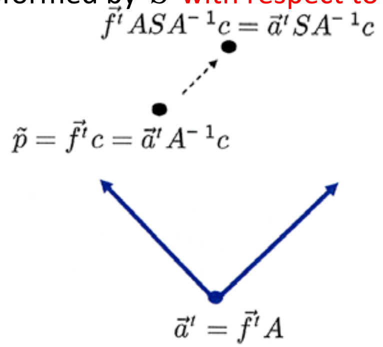
$\tilde{p}$ is transformed by S with respect to $\vec{\mathbf{f}}^t$



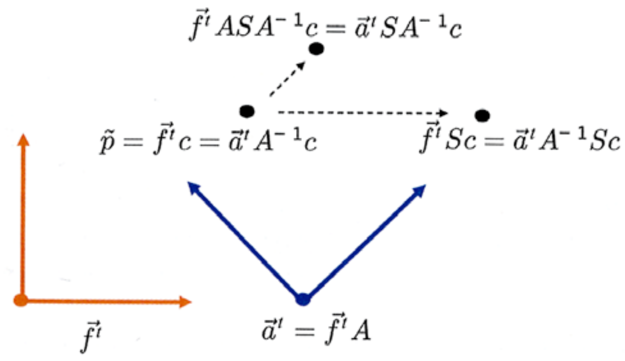
Scaling a point over frame

$$\tilde{p} = \vec{\mathbf{a}}^t A^{-1}\mathbf{c} \Rightarrow \vec{\mathbf{a}}^t S A^{-1}\mathbf{c}$$

$\tilde{p}$ is transformed by S with respect to $\vec{\mathbf{a}}^t$



Scaling a point over frame



Rotating a point over frame

- The same reasoning to transformations of frames themselves:

  $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t R$

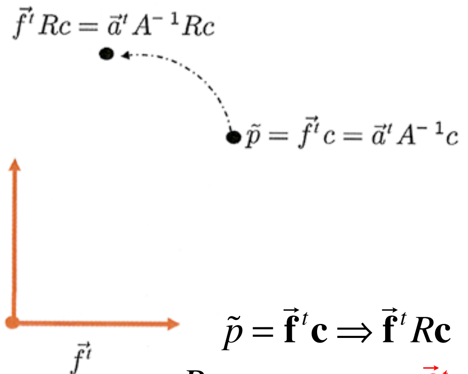
  $\vec{\mathbf{f}}^t$ is transformed by R with respect to $\vec{\mathbf{f}}^t$

- In another frame:

  $\vec{\mathbf{f}}^t = \vec{\mathbf{a}}^t A^{-1} \Rightarrow \vec{\mathbf{a}}^t R A^{-1}$

  $\vec{\mathbf{f}}^t$ is transformed by R with respect to $\vec{\mathbf{a}}^t$

Rotating a point over frame



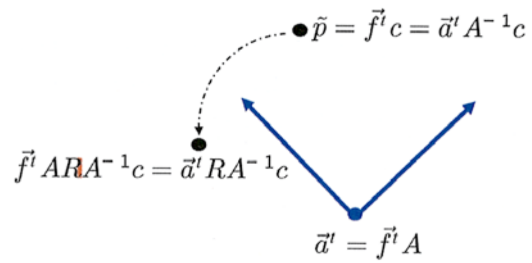
$\tilde{p} = \vec{\mathbf{f}}^t \mathbf{c} \Rightarrow \vec{\mathbf{f}}^t R\mathbf{c}$

$\tilde{p}$ is transformed by R with respect to $\vec{\mathbf{f}}^t$

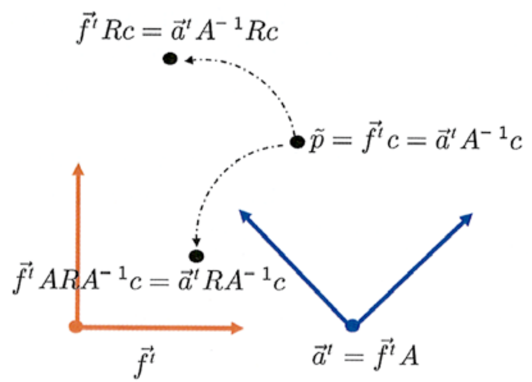
Rotating a point over frame

$\tilde{p} = \vec{\mathbf{a}}^t A^{-1}\mathbf{c} \Rightarrow \vec{\mathbf{a}}^t R A^{-1}\mathbf{c}$

$\tilde{p}$ is transformed by R with respect to $\vec{\mathbf{a}}^t$

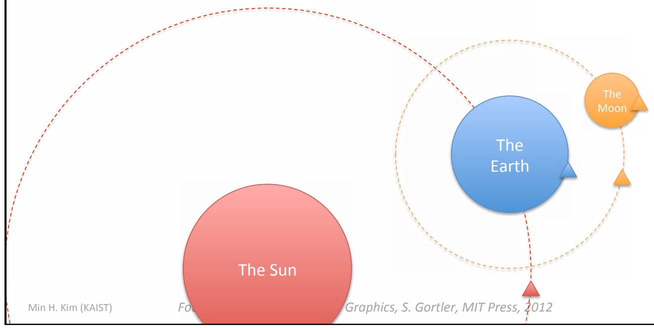


Rotating a point over frame



Auxiliary Frame

- You want to build the solar system
  - The Moon rotates around the Earth's frame
  - The Earth rotates around the Sun's frame



Transforms using an Auxiliary Frame

- Sometimes we need to transform a frame $\vec{\mathbf{f}}^t$ in some specific way, represented by a matrix M, with respect to some auxiliary frame $\vec{\mathbf{a}}^t$

  $\vec{\mathbf{a}}^t \Rightarrow \vec{\mathbf{f}}^t A$
- The transform frame can then be expressed as

  $\vec{\mathbf{f}}^t$

  $= \vec{\mathbf{a}}^t A^{-1}$

  $\Rightarrow \vec{\mathbf{a}}^t M A^{-1}$

  $= \vec{\mathbf{f}}^t A M A^{-1}$

Multiple Transformations

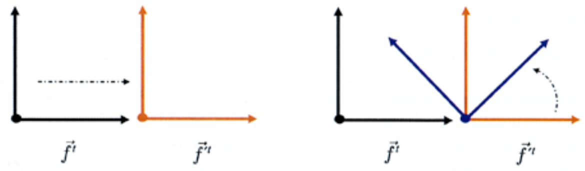
- Rotation and translation with frame

  $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t TR$
- NB in general, matrix multiplication is not commutative!!!

  $\vec{\mathbf{f}}^t TR \neq \vec{\mathbf{f}}^t RT$
- There are two different ways to apply multiple transformations
  - Local transformation
  - Global transformation

Local Transformations

- Local transformations $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t TR$



(a) Local translation (a) Local rotation

- In the first step,

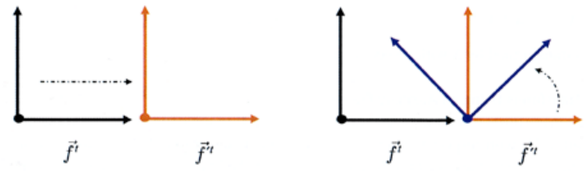
  $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t T = \vec{\mathbf{f}}^{\prime t}$

  $\vec{\mathbf{f}}^t$ is transformed by T with respect to $\vec{\mathbf{f}}^t$

  as the resulting frame: $\vec{\mathbf{f}}^{\prime t}$

Local Transformations

- Local transformations $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t TR$



(a) Local translation (a) Local rotation

- In the second step,

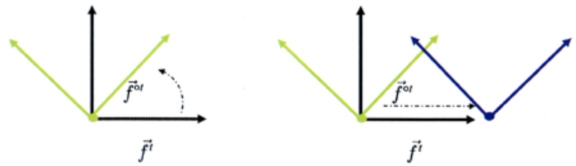
  $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t TR,$

  $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^{\prime t} R.$

  $\vec{\mathbf{f}}^t$ is transformed by R with respect to $\vec{\mathbf{f}}^{\prime t}$

Global Transformations

- Global transformations $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t TR$



(c) Global rotation (c) Global translation

- In the first step (in the reverse order)

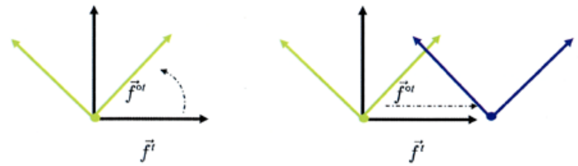
  $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t R = \vec{\mathbf{f}}^{\circ t}$

  $\vec{\mathbf{f}}^t$ is transformed by R with respect to $\vec{\mathbf{f}}^t$

  as the resulting frame: $\vec{\mathbf{f}}^{\circ t}$

Global Transformations

- Global transformations $\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t TR$



(c) Global rotation (c) Global translation

- In the second step

  $\vec{\mathbf{f}}^{\circ t} = \vec{\mathbf{f}}^t R \Rightarrow \vec{\mathbf{f}}^t TR$

  $\vec{\mathbf{f}}^{\circ t}$ is transformed by T with respect to $\vec{\mathbf{f}}^t$

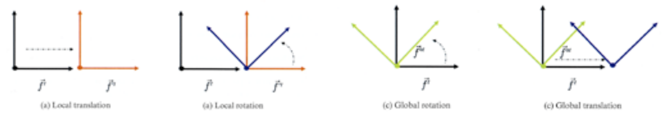
Two interpretations of transformations

- Two different ways for multiple transformations:
  1. (Local transformations) Translate with respect to $\vec{\mathbf{f}}^t$ then rotate with respect to the intermediate frame $\vec{\mathbf{f}}'^t$
  2. (Global transformations) Rotate with respect to $\vec{\mathbf{f}}^t$ then translate with respect to the original frame $\vec{\mathbf{f}}^t$

$$\vec{\mathbf{f}}^t \Rightarrow \vec{\mathbf{f}}^t TR$$

Local transformations

Global transformations



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